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UNIQUE SOLVABILITY OF CRACK PROBLEM WITH TIME–DEPENDENT FRICTION CONDITION IN LINEARIZED ELASTODYNAMIC BODY

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Abstract. This study considers a crack problem with a time-dependent friction condition in a linearized elastodynamic body. We suppose that the crack is fixed and the frictional force acting on the crack is given and depends on the time as well as space variables. The problem is then reduced to a variational inequality of the hyperbolic type. The unique existence of a solution is proved by using Galerkin's method and deriving *a priori* estimates for the penalized problem.

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Dedicated to Professor I. E. Egorov
on the occasion of his 70th birthday

1. Introduction

Friction problems have received immense attention since the renaissance period by Leonardo da Vinci, and are now recognized as a *tribology* discipline in material sciences. Friction arises because of the relative motion between two surfaces of bodies in contact. In mathematical models, such problems are described as the governing equation for the body motion and deformation, together with conditions that model the contact and friction forces. In addition, cracks in elastic bodies have been extensively studied because the appearance of a crack in a solid may lead to damage and even complete destruction. A contact condition for cracks is the so-called nonpenetration condition defined as a unilateral condition on the body displacement in order to exclude nonphysical phenomena such as the mutual penetration of the crack faces [1]. The main difficulty involved in crack problems is the allowance of the existence of singular solutions, caused by the lack of smoothness of domain.

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Friction conditions are phenomenologically derived as empirical laws. One of the most popular conditions is the Coulomb friction law in which slip occurs between two bodies, if the normal component of the contact force is proportional to the tangential component. The factor of proportionality is called the coefficient of friction, which characterizes the interface condition of the two bodies in contact. If the normal component in the Coulomb friction law is assumed to be a given function, the condition is called Tresca friction. From a mathematical point of view, the *static* contact problem with friction has been well studied [2–10]. [10] presents the existence theorem for a weak solution of an interfacial crack problem, where the crack is between two bonded dissimilar linearized elastic bodies, and the Coulomb friction law and nonpenetration condition are assumed to hold for the entire crack surface. Moreover, convergent series expansions of the solution near the crack tip have been derived using the Goursat–Kolosov–Muskhelishvili stress functions.

In contrast, analysis of the *dynamic* contact problem with friction is mathematically difficult because of the hyperbolic nature of the elastodynamic equation, and certain problems such as general contact problems or the Coulomb friction problem remain unsolved despite their practical importance, see [4] for an overview of the dynamic contact and friction problems. In the case of *viscoelastic* materials such as the classic Kelvin–Voigt viscoelastic model, the situation in problem gets better because regularity of the solution can be ensured utilizing the viscosity term. Therefore, the results for dynamic contact problems with or without friction have been obtained. However, the existence of a solution for the dynamic contact problem with the Coulomb friction law remains an open problem, even in the viscoelastic case. To address this difficulty, the contact condition has been proposed in velocities instead of displacements (see [4, Chapters 4, 5] and the references therein). As another method of overcoming this difficulty, approximation of the Coulomb friction law has been attempted by mollifying the traction in the friction term, which is a nonlocal condition also called averaged friction (cf. [11]). In [3], the Tresca friction problem without contact conditions for a linearized elastodynamic body, and not a viscoelastic one, has been considered, and a unique existence theorem has been established under the assumption that the given friction g does not depend on time variable t.

In summary, several problems, particularly the dynamic ones, pertaining to contact, friction, and cracks remain unsolved theoretically. The solutions to such problems may be applied in various fields such as fault rupture in earthquakes (cf. [12]) as well as in fluid dynamics (cf. [13]). In this study, we consider the Tresca friction problem for a fixed crack in an elastodynamic body, which is regarded as an extension of the result reported in [3] to a crack problem, where the threshold g may depend on t. The method for solving this problem is based on the method used in the proof of [3, Theorem 5.7 in Chapter I]. However, there are certain errors in [3], leading to an unnecessary requirement, namely, $\partial g/\partial t = 0$. The main objective of this study is to provide a corrected version with rigorous proof and apply it to the crack problem. The remainder of this paper is organized as follows. Section 2 formulates the initial-boundary value problem and introduces the variational formulation. Further, the equivalence of two formulations is discussed, and the main result is stated. In Section 3, our main theorem is proved with several steps. We first regularize the problem using penalization and approximate it using the Galerkin's method. We then establish three types of *a priori* estimates and prove the existence of a solution to the regularized problem. Next, we take the limit with respect to the regularized parameter and conclude the existence of the desired solution in a strong sense, which implies that the second time derivative of the solution exists in a usual sense rather than a distribution. Finally, we demonstrate the uniqueness of the solution.

2. Problem formulation and main result

2.1. Problem formulation. Let Ω be a bounded domain in $\mathbb{R}^3$, represented as an isotropic, homogeneous, and linearized elasticity. Let its boundary $\partial\Omega$ be a Lipschitz boundary with an outward normal vector $\boldsymbol{n} = (n_i)_{i=1,2,3}$ comprising mutually disjoint Neumann and Dirichlet parts Γ_{N} and Γ_{D}, respectively, such that $\partial\Omega = \overline{\Gamma_{\mathrm{N}}} \cup \overline{\Gamma_{\mathrm{D}}}$ and $\overline{\Gamma_{\mathrm{D}}} \neq \varnothing$. Let Γ be a two-dimensional oriented open manifold that splits Ω into two domains Ω_+ and Ω_- such that $\Gamma = \partial\Omega_+ \cap \partial\Omega_-$. We assume that boundaries $\partial\Omega_+$ and $\partial\Omega_-$ satisfy the Lipschitz condition, and $\partial\Omega_+ \cap \Gamma_{\mathrm{D}} \neq \varnothing$, $\partial\Omega_- \cap \Gamma_{\mathrm{D}} \neq \varnothing$. A crack is denoted by Γ_{C}, which is an open subset of Γ, i.e., $\overline{\Gamma_{\mathrm{C}}} \subset \Gamma$. In this study, we treat only the case of a fixed crack without propagation. The two crack faces $\Gamma_{\mathrm{C}}^{\pm}$ can be distinguished as the corresponding parts of $\partial\Omega_{\pm}$, and the normal vector $\boldsymbol{n}$ is selected such that the positive side Γ_{C}^{+} of Γ_{C} is in the direction inward to Ω_+ and the negative side Γ_{C}^{-} of Γ_{C} is in the direction outward to Ω_-. In addition, we denote the tangential unit vector on Γ_{C} by $\boldsymbol{\tau}$; Figure 1 illustrates the domain geometry. We refer to geometric set $\Omega_c = \Omega \setminus \overline{\Gamma_{\mathrm{C}}}$ obeying boundary $\partial\Omega \cup \Gamma_{\mathrm{C}}^{+} \cup \Gamma_{\mathrm{C}}^{-}$ as the "domain with crack" (Fig. 1). For a fixed final time $T > 0$, the time-space cylinder is denoted by $Q_c^T = (0, T) \times \Omega_c$.

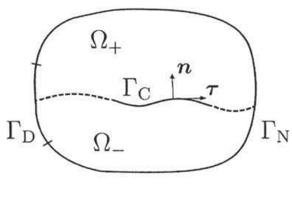

Fig 1. Domain.

The displacement vector is denoted by $\boldsymbol{u} = (u_i(t, \boldsymbol{x}))_{i=1,2,3}$

$$\boldsymbol{u} = \begin{cases} \boldsymbol{u}^+ = (u_i^+(t, \boldsymbol{x}))_{i=1,2,3} & \text{in } (0, T) \times \Omega_+, \\ \boldsymbol{u}^- = (u_i^-(t, \boldsymbol{x}))_{i=1,2,3} & \text{in } (0, T) \times \Omega_-, \end{cases}$$

and the linearized strain tensor by $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))_{i,j=1,2,3}$ given by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}}), \tag{1}$$

where superscript T denotes the transposition. Stress tensor $\boldsymbol{\sigma}(\boldsymbol{u}) = (\sigma_{ij}(\boldsymbol{u}))_{i,j=1,2,3}$ is described according to Hooke's law as

$$\boldsymbol{\sigma}(\boldsymbol{u}) = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda(\operatorname{tr}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{I}, \tag{2}$$

where $\boldsymbol{I}$ is the identity tensor, and λ and μ are Lamé constants satisfying $\mu > 0$ and $3\lambda + 2\mu > 0$. Based on Newton's law, the balance of linear momentum is expressed as

$$\rho\boldsymbol{u}_{tt} = \nabla\cdot\boldsymbol{\sigma} + \rho\boldsymbol{f}, \tag{3}$$

where $\rho > 0$ is the (constant) density and $\boldsymbol{f}$ is the specific body force. In the following, we denote the velocity and acceleration vectors by $\boldsymbol{u}' = \boldsymbol{u}_t$ and $\boldsymbol{u}'' = \boldsymbol{u}_{tt}$, respectively; i.e., $'$ denotes the time derivative.

Summing (1)–(3) provides a system of linearized elasticity equations for $\boldsymbol{u}$, as follows:

$$\rho\boldsymbol{u}'' - \mu\triangle\boldsymbol{u} - (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) = \rho\boldsymbol{f} \quad \text{in} \quad Q_c^T. \tag{4}$$

In this study, we consider the following initial-boundary value problem (E), which is an elastodynamic problem with the Tresca friction condition on Γ_{C}; specifically, the frictional force is given:

$$(E)\quad \begin{cases} \rho\boldsymbol{u}'' - \mu\triangle\boldsymbol{u} - (\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) = \rho\boldsymbol{f} & \text{in } Q_c^T, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } (0,T)\times\Gamma_{\mathrm{D}}, \\ \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{F} & \text{on } (0,T)\times\Gamma_{\mathrm{N}}, \\ \sigma_n = F_n & \text{on } (0,T)\times\Gamma_{\mathrm{C}}, \\ [\![\boldsymbol{\sigma}_\tau]\!] = \boldsymbol{0}, \quad |\boldsymbol{\sigma}_\tau| \le g, \quad [\![\boldsymbol{u}'_\tau]\!]\cdot\boldsymbol{\sigma}_\tau = g|[\![\boldsymbol{u}'_\tau]\!]| & \text{on } (0,T)\times\Gamma_{\mathrm{C}}, \\ \boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_0, \quad \boldsymbol{u}_t(0,\boldsymbol{x}) = \dot{\boldsymbol{u}}_0 & \text{in } \Omega_c. \end{cases}$$

Here, surface traction $\boldsymbol{F} = \boldsymbol{F}(t,\boldsymbol{x})$ is prescribed on $(0,T)\times\Gamma_{\mathrm{N}}$, $g = g(t,\boldsymbol{x})$ is the frictional force acting on Γ_{C} in the form of $g = \mathscr{F}|F_n|$ with a given $F_n = F_n(t,\boldsymbol{x})$ and frictional coefficient $\mathscr{F} \ge 0$. We denote the jump across the crack by $[\![\cdot]\!] = \cdot|_{\Gamma_{\mathrm{C}}^+} - \cdot|_{\Gamma_{\mathrm{C}}^-}$. The normal and tangential components of the vectors at the boundary are expressed by the following notations:

$$u_n = \boldsymbol{u}\cdot\boldsymbol{n}, \quad \boldsymbol{u}_\tau = \boldsymbol{u} - u_n\boldsymbol{n}, \quad \sigma_n = (\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n})\cdot\boldsymbol{n}, \quad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} - \sigma_n\boldsymbol{n}.$$

Since $\boldsymbol{u}_\tau\cdot\boldsymbol{n} = \boldsymbol{\sigma}_\tau\cdot\boldsymbol{n} = 0$, we have

$$(\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n})\cdot\boldsymbol{v} = (\sigma_n\boldsymbol{n} + \boldsymbol{\sigma}_\tau)\cdot(v_n\boldsymbol{n} + \boldsymbol{v}_\tau) = \sigma_n v_n + \boldsymbol{\sigma}_\tau\cdot\boldsymbol{v}_\tau. \tag{5}$$

2.2. Variational formulation of the problem (E). We introduce function spaces $V = \{\boldsymbol{v} \in H^1(\Omega_c) \mid \boldsymbol{v} = 0 \text{ on } \Gamma_{\mathrm{D}}\}$ and $H = L^2(\Omega_c)$ equipped with an inner product denoted by $(\cdot,\cdot)$. The norm in V is defined as

$$\|\boldsymbol{u}\|_V^2 = \|\boldsymbol{u}^+\|_{H^1(\Omega_+)}^2 + \|\boldsymbol{u}^-\|_{H^1(\Omega_-)}^2.$$

We can similarly define the norm in H. $(\cdot,\cdot)_{\Gamma_{\mathrm{N}}}$ and $(\cdot,\cdot)_{\Gamma_{\mathrm{C}}}$ represent the duality between Lions–Magenes spaces $H_{00}^{1/2}(\Gamma_{\mathrm{N}})$ and $H_{00}^{1/2}(\Gamma_{\mathrm{C}})$ of functions that can be extended by zero and their dual, respectively (see, for example, [14, Theorem 11.7,

p. 66] and [1, Section 1.4]). Let a be a bilinear continuous symmetric form defined as

$$a(\boldsymbol{u},\boldsymbol{v}) = \int\limits_{\Omega_c} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\,d\boldsymbol{x} = \int\limits_{\Omega_c} \frac{\mu}{2}(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^{\mathrm{T}}) : (\nabla\boldsymbol{v}+(\nabla\boldsymbol{v})^{\mathrm{T}})+\lambda(\nabla\cdot\boldsymbol{u})(\nabla\cdot\boldsymbol{v})\,d\boldsymbol{x}.$$

For all $\boldsymbol{u}$, $\boldsymbol{v} \in H^1(\Omega_c)$, where $\boldsymbol{M} : \boldsymbol{n} = (\operatorname{tr}\boldsymbol{M}^{\mathrm{T}}\boldsymbol{n}) = (\operatorname{tr}\boldsymbol{N}^{\mathrm{T}}\boldsymbol{M})$. It follows from Korn's inequality that there exists a positive constant $c_K > 0$ such that

$$a(\boldsymbol{u},\boldsymbol{u}) \geq c_K\|\boldsymbol{u}\|_V^2 \quad \forall \boldsymbol{u}\in V, \tag{6}$$

refer for the details to [3, Theorem 3.3, p. 115]. We now formulate problem (E) in the variational sense.

Problem (E'). *Assume that*

$$\boldsymbol{u}_0 = \boldsymbol{u}_0(\boldsymbol{x}) \in V, \quad \dot{\boldsymbol{u}}_0 = \dot{\boldsymbol{u}}_0(\boldsymbol{x}) \in V, \quad \boldsymbol{f} = \boldsymbol{f}(t,\boldsymbol{x}) \in H^1(0,T;H),$$

$$\boldsymbol{F} \in H^2(0,T;L^2(\Gamma_{\mathrm{N}})), \quad F_n \in H^2(0,T;L^2(\Gamma_{\mathrm{C}})), \quad g = g(t,\boldsymbol{x}) \in H^2(0,T;L^2(\Gamma_{\mathrm{C}})).$$

Find $\boldsymbol{u}$ satisfying

$$\boldsymbol{u} \in L^\infty(0,T;V), \quad \boldsymbol{u}' \in L^\infty(0,T;V), \quad \boldsymbol{u}'' \in L^\infty(0,T;H), \tag{7}$$

such that for all $\boldsymbol{v}\in V$ and for a.e. $t \in (0,T)$ the variational inequality

$$\rho(\boldsymbol{u}''(t),\boldsymbol{v}-\boldsymbol{u}'(t)) + a(\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}'(t)) + (g(t),|[\![\boldsymbol{v}_\tau]\!]|)_{\Gamma_{\mathrm{C}}} - (g(t),|[\![\boldsymbol{u}'_\tau(t)]\!]|)_{\Gamma_{\mathrm{C}}} \geq \langle \boldsymbol{L}(t),\boldsymbol{v}-\boldsymbol{u}'(t)\rangle \tag{8}$$

and the initial conditions

$$\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_0, \quad \boldsymbol{u}'(0,\boldsymbol{x}) = \dot{\boldsymbol{u}}_0 \quad \forall \boldsymbol{x}\in\Omega_c \tag{9}$$

hold, where $\boldsymbol{L}\in L^2(0,T;V')$ is given by

$$\langle \boldsymbol{L}(t),\boldsymbol{v}\rangle = \rho(\boldsymbol{f}(t),\boldsymbol{v}) + (\boldsymbol{F}(t),\boldsymbol{v})_{\Gamma_{\mathrm{N}}} - (F_n(t),[\![v_n]\!])_{\Gamma_{\mathrm{C}}}.$$

Next, we present the equivalence of problems (E) and (E'). We assume that $\boldsymbol{u}$ satisfies (E) in the classical sense. Taking the scalar product of (4) with $\boldsymbol{v}-\boldsymbol{u}'$ ($\boldsymbol{v}\in V$), and using Green's formula and (5) we obtain

$$\begin{aligned}\rho(\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}'(t)) &= \rho(\boldsymbol{u}''(t),\boldsymbol{v}-\boldsymbol{u}'(t)) - (\nabla\cdot\boldsymbol{\sigma}(t),\boldsymbol{v}-\boldsymbol{u}'(t))\\ &= \rho(\boldsymbol{u}''(t),\boldsymbol{v}-\boldsymbol{u}'(t)) + a(\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}'(t)) - (\boldsymbol{F}(t),\boldsymbol{v}-\boldsymbol{u}'(t))_{\Gamma_{\mathrm{N}}}\\ &\qquad + (F_n(t),[\![v_n]\!]-[\![u'_n(t)]\!])_{\Gamma_{\mathrm{C}}} + (\boldsymbol{\sigma}_\tau(t),[\![\boldsymbol{v}_\tau]\!]-[\![\boldsymbol{u}'_\tau(t)]\!])_{\Gamma_{\mathrm{C}}}.\end{aligned}$$

From the conditions on Γ_{C} in (E) one can see that

$$(\boldsymbol{\sigma}_\tau(t),[\![\boldsymbol{v}_\tau]\!]-[\![\boldsymbol{u}'_\tau(t)]\!])_{\Gamma_{\mathrm{C}}} \leq (g(t),|[\![\boldsymbol{v}_\tau]\!]|)_{\Gamma_{\mathrm{C}}} - (g(t),|[\![\boldsymbol{u}'_\tau(t)]\!]|)_{\Gamma_{\mathrm{C}}}.$$

Consequently, solution $\boldsymbol{u}$ of (E) satisfies (8).

Conversely, if $\boldsymbol{u}$ is a solution problem (E') and has additional regularity, by substituting $\boldsymbol{v} = \boldsymbol{u}' \pm \boldsymbol{\phi}$ ($\boldsymbol{\phi} \in C_0^\infty(\Omega_c)$) in (8) we obtain

$$\rho(\boldsymbol{u}''(t), \boldsymbol{\phi}) + a(\boldsymbol{u}(t), \boldsymbol{\phi}) = \rho(\boldsymbol{f}(t), \boldsymbol{\phi}) \quad \forall \boldsymbol{\phi} \in C_0^\infty(\Omega_c).$$

This implies that (4) holds in the distributional sense. The generalized Green's formula (e.g. [1]) yields

$$\begin{aligned} a(\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}'(t)) = &-(\nabla \cdot \boldsymbol{\sigma}, \boldsymbol{v} - \boldsymbol{u}'(t)) + (\boldsymbol{\sigma n}, \boldsymbol{v} - \boldsymbol{u}'(t))_{\Gamma_{\mathrm{N}}} \\ &- (\boldsymbol{\sigma}(\boldsymbol{u}^+)\boldsymbol{n}, \boldsymbol{v}^+ - (\boldsymbol{u}^+)'(t))_{\Gamma_{\mathrm{C}}} + (\boldsymbol{\sigma}(\boldsymbol{u}^-)\boldsymbol{n}, \boldsymbol{v}^- - (\boldsymbol{u}^-)'(t))_{\Gamma_{\mathrm{C}}}. \end{aligned}$$

Therefore, we can define σ_n and $\boldsymbol{\sigma}_\tau$ in the dual space of $H_{00}^{1/2}$. Then, it follows from (8) and (9) that

$$\begin{aligned} &(\boldsymbol{\sigma}(t)\boldsymbol{n} - \boldsymbol{F}(t), \boldsymbol{v} - \boldsymbol{u}'(t))_{\Gamma_{\mathrm{N}}} + (g(t), |[\![\boldsymbol{v}_\tau]\!]| - |[\![\boldsymbol{u}'_\tau]\!](t)|)_{\Gamma_{\mathrm{C}}} \\ &\quad - (\boldsymbol{\sigma}_\tau^+(t), \boldsymbol{v}_\tau^+ - (\boldsymbol{u}^+)'_\tau(t))_{\Gamma_{\mathrm{C}}} + (\boldsymbol{\sigma}_\tau^-(t), \boldsymbol{v}_\tau^- - (\boldsymbol{u}^-)'_\tau(t))_{\Gamma_{\mathrm{C}}} \\ &\quad - (\sigma_n^+(t) - F_n(t), v_n^+ - (u^+)'_n(t))_{\Gamma_{\mathrm{C}}} + (\sigma_n^-(t) - F_n(t), v_n^- - (u^-)'_n(t))_{\Gamma_{\mathrm{C}}} \geq 0 \end{aligned}$$

for all $\boldsymbol{v} \in V$, where $\sigma_n^\pm = (\boldsymbol{\sigma}(\boldsymbol{u}^\pm)\boldsymbol{n}) \cdot \boldsymbol{n}$ and $\boldsymbol{\sigma}_\tau^\pm = \boldsymbol{\sigma}(\boldsymbol{u}^\pm)\boldsymbol{n} - \sigma_n^\pm \boldsymbol{n}$. Considering a special $\boldsymbol{v}$ such as $\boldsymbol{v}^\pm = (\boldsymbol{u}^\pm)'$ on Γ_{C} implies that $\boldsymbol{\sigma n} = \boldsymbol{F}$ on Γ_{N}; considering $\boldsymbol{v}$ as $\boldsymbol{v} = \boldsymbol{u}'$ on Γ_{N} and $\boldsymbol{v}_\tau^\pm = (\boldsymbol{u}^\pm)'_\tau$ on Γ_{C} gives $\sigma_n = F_n$ on Γ_{C}. Furthermore, substituting $\boldsymbol{v} = \boldsymbol{u}' \pm \boldsymbol{\phi}$ ($\boldsymbol{\phi} \in H_0^1(\Omega)$) we obtain

$$([\![\boldsymbol{\sigma}_\tau(\boldsymbol{u})]\!], \boldsymbol{\phi}_\tau)_{\Gamma_{\mathrm{C}}} = 0,$$

which indicates that $[\![\boldsymbol{\sigma}_\tau]\!] = \boldsymbol{0}$. Hence,

$$(g(t), |[\![\boldsymbol{v}_\tau]\!]| - |[\![\boldsymbol{u}'_\tau(t)]\!]|)_{\Gamma_{\mathrm{C}}} - (\boldsymbol{\sigma}_\tau(t), [\![\boldsymbol{v}_\tau]\!] - [\![\boldsymbol{u}'_\tau(t)]\!])_{\Gamma_{\mathrm{C}}} \geq 0.$$

Next, we replace $\boldsymbol{v}_\tau$ with $\pm\beta\boldsymbol{v}_\tau$ in which constant $\beta \geq 0$. This gives that

$$\beta\{(g(t), |[\![\boldsymbol{v}_\tau]\!]|)_{\Gamma_{\mathrm{C}}} \mp (\boldsymbol{\sigma}_\tau(t), [\![\boldsymbol{v}_\tau]\!])_{\Gamma_{\mathrm{C}}}\} \geq (g(t), |[\![\boldsymbol{u}'_\tau(t)]\!]|)_{\Gamma_{\mathrm{C}}} - (\boldsymbol{\sigma}_\tau(t), [\![\boldsymbol{u}'_\tau(t)]\!])_{\Gamma_{\mathrm{C}}}.$$

The arbitrariness of β yields

$$(g(t), |[\![\boldsymbol{v}_\tau]\!]|)_{\Gamma_{\mathrm{C}}} \mp (\boldsymbol{\sigma}_\tau(t), [\![\boldsymbol{v}_\tau]\!])_{\Gamma_{\mathrm{C}}} \geq 0 \quad \forall \boldsymbol{v} \in V.$$

This inequality implies that

$$(g(t), |[\![\boldsymbol{\psi}]\!]|)_{\Gamma_{\mathrm{C}}} \mp (\boldsymbol{\sigma}_\tau(t), [\![\boldsymbol{\psi}]\!])_{\Gamma_{\mathrm{C}}} \geq 0 \quad \forall \boldsymbol{\psi} \in H_{00}^{1/2}(\Gamma_{\mathrm{C}}) \text{ with } \psi_n = 0,$$

namely, $|\boldsymbol{\sigma}_\tau| \leq g$ a.e. on $(0, T) \times \Gamma_{\mathrm{C}}$. From this we obtain

$$(g(t), |[\![\boldsymbol{u}'_\tau(t)]\!]|)_{\Gamma_{\mathrm{C}}} - (\boldsymbol{\sigma}_\tau(t), [\![\boldsymbol{u}'_\tau(t)]\!])_{\Gamma_{\mathrm{C}}} = 0,$$

which implies that $g|[\![\boldsymbol{u}'_\tau]\!]| = \boldsymbol{\sigma}_\tau \cdot [\![\boldsymbol{u}'_\tau]\!]$ a.e. on $(0, T) \times \Gamma_{\mathrm{C}}$.

2.3. Main result. Theorem 5.7 [3, p. 156] shows the unique existence of a solution $\boldsymbol{u}$ for problem (E') in the case where g does not depend on t. Further, we extend the result in [3] to the case where g depends on t and $\boldsymbol{x}$.

Theorem. *Assume that the initial data for* $\boldsymbol{u}$ *and* $\boldsymbol{u}'$ *satisfy the following compatibility conditions at* $t = 0$*:*

$$\begin{cases} -\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{u}_0) \in H, & \\ \boldsymbol{\sigma}(\boldsymbol{u}_0)\boldsymbol{n} = \boldsymbol{F}(0) & \text{on } \Gamma_{\mathrm{N}}, \\ \sigma_n(\boldsymbol{u}_0^+) = \sigma_n(\boldsymbol{u}_0^-) = F_n(0) & \text{on } \Gamma_{\mathrm{C}}, \\ \boldsymbol{\sigma}_\tau(\boldsymbol{u}_0^+) = \boldsymbol{\sigma}_\tau(\boldsymbol{u}_0^-) = \boldsymbol{0}, \dot{\boldsymbol{u}}_{0\tau} = \boldsymbol{0} & \text{on } \Gamma_{\mathrm{C}}. \end{cases} \tag{10}$$

Then there exists a unique solution $\boldsymbol{u} \in W^{2,\infty}(0,T;H) \cap W^{1,\infty}(0,T;V)$ *for problem* (E').

Remark 1. (i) The compatibility conditions in the fourth line are stronger than just requiring that $\boldsymbol{u}_0$ and $\dot{\boldsymbol{u}}_0$ satisfy the friction condition. We are unclear whether this can be relaxed.

(ii) For a linear wave equation, the class of commonly used weak solutions is $W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V)$ (see, for example, [15, Section 7.2]). However, this class is inappropriate for (8) because the trace of $\boldsymbol{u}'$ appearing there would be undefined.

(iii) The following characterization of $H^1(\Omega_c)$ is required when constructing the Galerkin approximation shown below:

$$H^1(\Omega_c) = \{(\boldsymbol{v}^+, \boldsymbol{v}^-) \in H^1(\Omega^+) \times H^1(\Omega^-) : \boldsymbol{v}^+ = \boldsymbol{v}^- \text{ on } \Gamma \setminus \Gamma_{\mathrm{C}}\}, \tag{11}$$

which is particularly separable as a closed subspace of $H^1(\Omega^+) \times H^1(\Omega^-)$.

3. Proof of Theorem

In this section, we present the proof of theorem.

3.1. Regularized problem. For $\epsilon > 0$, we introduce

$$\varphi_\epsilon(\boldsymbol{x}) = \sqrt{|\boldsymbol{x}|^2 + \epsilon^2}, \quad \boldsymbol{\alpha}_\epsilon(\boldsymbol{x}) = \nabla\varphi_\epsilon(\boldsymbol{x})$$

and attempt to construct a solution $\boldsymbol{u} \in W^{2,\infty}(0,T;H) \cap W^{1,\infty}(0,T;V)$ such that

$$\rho(\boldsymbol{u}''(t), \boldsymbol{v} - \boldsymbol{u}'(t)) + a(\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}'(t)) + (g(t), \varphi_\epsilon([\![\boldsymbol{v}_\tau]\!])_{\Gamma_{\mathrm{C}}} - (g(t), \varphi_\epsilon([\![\boldsymbol{u}'_\tau(t)]\!])_{\Gamma_{\mathrm{C}}} \\ \geq \langle \boldsymbol{L}(t), \boldsymbol{v} - \boldsymbol{u}'(t)\rangle \quad \forall \boldsymbol{v} \in V, \quad \text{a.e. } t \in (0,T), \tag{12}$$

and $\boldsymbol{u}(0) = \boldsymbol{u}_0$ and $\boldsymbol{u}'(0) = \dot{\boldsymbol{u}}_0$.

Since φ_ϵ is differentiable, the above variational inequality is equivalent to

$$\rho(\boldsymbol{u}''(t), \boldsymbol{v}) + a(\boldsymbol{u}(t), \boldsymbol{v}) + (g(t)\boldsymbol{\alpha}_\epsilon([\![\boldsymbol{u}'_\tau(t)]\!]), [\![\boldsymbol{v}_\tau]\!])_{\Gamma_{\mathrm{C}}} = \langle \boldsymbol{L}(t), \boldsymbol{v}\rangle \; \forall \boldsymbol{v} \in V, \text{ a.e. } t \in (0,T). \tag{13}$$

Indeed, considering $\boldsymbol{v} = \boldsymbol{w} - \boldsymbol{u}'$ ($\boldsymbol{w} \in V$) in (13) leads to (12) by virtue of the convexity of φ_ϵ. On the other hand, we substitute test functions $\boldsymbol{v} = \boldsymbol{u}' \pm \beta\boldsymbol{w}$ ($\boldsymbol{w} \in V$) in (12) and then taking limit $\beta \to 0$ leads to (13).

3.2. Galerkin approximation problem. We consider the Galerkin approximation to solve (13). Since $V \subset H^1(\Omega_c)$ is separable (recall Remark 1(iii)), there exist countable members $\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots \in V$, which are linearly independent on each other, such that

$$\overline{\bigcup_{m=1}^{\infty} V_m} = V, \quad V_m = \mathrm{span}\{\boldsymbol{w}_k\}_{k=1}^m. \tag{14}$$

We assume that $\boldsymbol{u}_0, \dot{\boldsymbol{u}}_0 \in V_m$; otherwise, $\boldsymbol{u}_0$ and $\dot{\boldsymbol{u}}_0$ should be added to members $\{\boldsymbol{w}_k\}_{k=1}^m$.

For a fixed positive integer m, the approximate problem involves determining $c_k(t)\,(k = 1, \ldots, m)$ such that $\boldsymbol{u}_m = \sum_{k=1}^m c_k(t)\boldsymbol{w}_k(\boldsymbol{x})$ satisfies

$$\rho(\boldsymbol{u}_m''(t), \boldsymbol{w}_k) + a(\boldsymbol{u}_m(t), \boldsymbol{w}_k) + (g(t)\boldsymbol{\alpha}_\epsilon([\![\boldsymbol{u}_{m\tau}'(t)]\!]), [\![\boldsymbol{w}_{k\tau}]\!])_{\Gamma_{\mathrm{C}}} = \langle \boldsymbol{L}(t), \boldsymbol{w}_k\rangle \quad k = 1, \ldots, m, \ \forall t \in (0, T), \tag{15}$$

and the initial conditions $\boldsymbol{u}_m(0) = \boldsymbol{u}_0$ and $\boldsymbol{u}_m'(0) = \dot{\boldsymbol{u}}_0$.

This is a finite-dimensional system of ODEs that admits a local-in-time unique solution $\{c_k(t) \in C^3([0, \tilde{T}])\}_{k=1}^m$ for certain $0 < \tilde{T} \leq T$. Because the *a priori* estimates below imply that $\tilde{T}$ can be extended to T, we use T instead of $\tilde{T}$ from the beginning.

3.3. A priori estimate 1: $\boldsymbol{u}_m \in W^{1,\infty}(0, T; H) \cap L^\infty(0, T; V)$. Multiplying (15) by $c_k'(t)$ and taking the summation for $k = 1, \ldots, m$, we obtain

$$\rho(\boldsymbol{u}_m''(t), \boldsymbol{u}_m'(t)) + a(\boldsymbol{u}_m(t), \boldsymbol{u}_m'(t)) + (g(t)\boldsymbol{\alpha}_\epsilon([\![\boldsymbol{u}_{m\tau}'(t)]\!]), [\![\boldsymbol{u}_{m\tau}'(t)]\!])_{\Gamma_{\mathrm{C}}} = \langle \boldsymbol{L}(t), \boldsymbol{u}_m'(t)\rangle$$

for all $t \in (0, T)$. Since the third term on the left-hand side is nonnegative, one has

$$\frac{1}{2}\frac{d}{dt}\big(\rho\|\boldsymbol{u}_m'(t)\|_H^2 + a(\boldsymbol{u}_m(t), \boldsymbol{u}_m(t))\big) \leq \rho(\boldsymbol{f}(t), \boldsymbol{u}_m'(t)) + (\boldsymbol{F}(t), \boldsymbol{u}_m'(t))_{\Gamma_{\mathrm{N}}} - (F_n(t), [\![u_{mn}'(t)]\!])_{\Gamma_{\mathrm{C}}}.$$

Integration with respect to t gives

$$\begin{aligned}
&\frac{1}{2}\big(\rho\|\boldsymbol{u}_m'(t)\|_H^2 + a(\boldsymbol{u}_m(t), \boldsymbol{u}_m(t))\big) - \frac{1}{2}\big(\rho\|\boldsymbol{u}_m'(0)\|_H^2 + a(\boldsymbol{u}_m(0), \boldsymbol{u}_m(0))\big)\\
&\quad\leq \rho\int_0^t (\boldsymbol{f}(s), \boldsymbol{u}_m'(s))\,ds + \int_0^t (\boldsymbol{F}(s), \boldsymbol{u}_m'(s))_{\Gamma_{\mathrm{N}}}\,ds - \int_0^t (F_n(s), [\![\boldsymbol{u}_{mn}'(s)]\!])_{\Gamma_{\mathrm{C}}}\,ds\\
&\quad= \rho\int_0^t (\boldsymbol{f}(s), \boldsymbol{u}_m'(s))\,ds - \int_0^t (\boldsymbol{F}'(s), \boldsymbol{u}_m(s))_{\Gamma_{\mathrm{N}}}\,ds + [(\boldsymbol{F}(s), \boldsymbol{u}_m(s))_{\Gamma_{\mathrm{N}}}]_{s=0}^{s=t}\\
&\qquad + \int_0^t (F_n'(s), [\![u_{mn}(s)]\!])_{\Gamma_{\mathrm{C}}}\,ds - [(F_n(s), [\![u_{mn}(s)]\!])_{\Gamma_{\mathrm{C}}}]_{s=0}^{s=t} \quad \forall t \in [0, T].
\end{aligned}$$

Using the initial conditions, Hölder's inequality, Korn's inequality (6), and the trace inequalities, we have

$$\begin{aligned}
&\|\boldsymbol{u}_m'(t)\|_H^2 + \|\boldsymbol{u}_m(t)\|_V^2\\
&\leq C(\|\dot{\boldsymbol{u}}_0\|_H^2 + \|\boldsymbol{u}_0\|_V^2 + \|\boldsymbol{f}\|_{L^2(0,T;H)}^2 + \|\boldsymbol{F}\|_{H^1(0,T;L^2(\Gamma_{\mathrm{N}}))}^2 + \|F_n\|_{H^1(0,T;L^2(\Gamma_{\mathrm{C}}))}^2)\\
&\qquad + C\int_0^t (\|\boldsymbol{u}_m'(s)\|_H^2 + \|\boldsymbol{u}_m(s)\|_V^2)\,ds.
\end{aligned}$$

Throughout this paper, we denote a generic positive constant by C. Then, by the Gronwall inequality, we obtain

$$\|\boldsymbol{u}'_m(t)\|_H^2 + \|\boldsymbol{u}_m(t)\|_V^2 \\ \le Ce^{Ct}\big(\|\dot{\boldsymbol{u}}_0\|_H^2 + \|\boldsymbol{u}_0\|_V^2 + \|\boldsymbol{f}\|^2_{L^2(0,T;H)} + \|\boldsymbol{F}\|^2_{H^1(0,T;L^2(\Gamma_{\rm N}))} + \|F_n\|^2_{H^1(0,T;L^2(\Gamma_{\rm C}))}\big).$$

3.4. A priori estimate 2: $\boldsymbol{u}'_m \in W^{1,\infty}(0,T;H)\cap L^\infty(0,T;V)$. Differentiating (15) with respect to t, we have

$$\rho(\boldsymbol{u}'''_m(t), \boldsymbol{w}_k) + a(\boldsymbol{u}'_m(t), \boldsymbol{w}_k) \\ + (g'(t)\boldsymbol{\alpha}_\epsilon([\![\boldsymbol{u}'_{m\tau}(t)]\!]), [\![\boldsymbol{w}_{k\tau}]\!])_{\Gamma_{\rm C}} + (g(t)(\nabla\boldsymbol{\alpha}_\epsilon|_{[\![\boldsymbol{u}'_{m\tau}(t)]\!]})[\![\boldsymbol{u}''_{m\tau}(t)]\!], [\![\boldsymbol{w}_{k\tau}]\!])_{\Gamma_{\rm C}} \\ = \rho(\boldsymbol{f}'(t), \boldsymbol{w}_k) + (\boldsymbol{F}'(t), \boldsymbol{w}_k)_{\Gamma_{\rm N}} - (F'_n(t), [\![w_n]\!])._{\Gamma_{\rm C}}, \quad (16)$$

where $\nabla\boldsymbol{\alpha}_\epsilon|_{\boldsymbol{x}} = \nabla\boldsymbol{\alpha}_\epsilon(\boldsymbol{x})$ is a semipositive definite $d\times d$ matrix for all $\boldsymbol{x}\in\mathbb{R}^d$ as a consequence of the convexity of $\varphi_\epsilon(\boldsymbol{x})$.

Multiplying (16) by $c''_k(t)$ and taking the summation for $k = 1,\dots,m$, we obtain

$$\rho(\boldsymbol{u}'''_m(t), \boldsymbol{u}''_m(t)) + a(\boldsymbol{u}'_m(t), \boldsymbol{u}''_m(t)) \\ + (g'(t)\boldsymbol{\alpha}_\epsilon([\![\boldsymbol{u}'_{m\tau}(t)]\!]), [\![\boldsymbol{u}''_{m\tau}(t)]\!])_{\Gamma_{\rm C}} + (g(t)(\nabla\boldsymbol{\alpha}_\epsilon|_{[\![\boldsymbol{u}'_{m\tau}(t)]\!]})[\![\boldsymbol{u}''_{m\tau}(t)]\!], [\![\boldsymbol{u}''_{m\tau}(t)]\!])_{\Gamma_{\rm C}} \\ = \rho(\boldsymbol{f}'(t), \boldsymbol{u}''_m(t)) + (\boldsymbol{F}'(t), \boldsymbol{u}''_m(t))_{\Gamma_{\rm N}} - (F'_n(t), [\![u''_{mn}(t)]\!])_{\Gamma_{\rm C}}.$$

Noting that the third term on the left-hand side is $(g'(t), \frac{d}{dt}\varphi_\epsilon([\![\boldsymbol{u}'_{m\tau}(t)]\!])_{\Gamma_{\rm C}}$ and that the fourth term is nonnegative, we deduce that

$$\frac{1}{2}\frac{d}{dt}\big(\rho\|\boldsymbol{u}''_m(t)\|_H^2 + a(\boldsymbol{u}'_m(t), \boldsymbol{u}'_m(t))\big) \le \rho(\boldsymbol{f}'(t), \boldsymbol{u}''_m(t)) \\ + (\boldsymbol{F}'(t), \boldsymbol{u}''_m(t))_{\Gamma_{\rm N}} - (F'_n(t), [\![u''_{mn}(t)]\!])_{\Gamma_{\rm C}} - \left(g'(t), \frac{d}{dt}\varphi_\epsilon([\![\boldsymbol{u}'_{m\tau}(t)]\!])\right)_{\Gamma_{\rm C}}.$$

Integration with respect to t gives

$$\frac{1}{2}\big(\rho\|\boldsymbol{u}''_m(t)\|_H^2 + a(\boldsymbol{u}'_m(t), \boldsymbol{u}'_m(t))\big) - \frac{1}{2}\big(\rho\|\boldsymbol{u}''_m(0)\|_H^2 + a(\boldsymbol{u}'_m(0), \boldsymbol{u}'_m(0))\big) \\ \le \rho\int_0^t (\boldsymbol{f}'(s), \boldsymbol{u}''_m(s))\,ds + \int_0^t (\boldsymbol{F}'(s), \boldsymbol{u}''_m(s))_{\Gamma_{\rm N}}\,ds - \int_0^t (F'_n(s), [\![u''_{mn}(s)]\!])_{\Gamma_{\rm C}}\,ds \\ - \int_0^t \left(g'(s), \frac{d}{ds}\varphi_\epsilon([\![\boldsymbol{u}'_{m\tau}(s)]\!])\right)_{\Gamma_{\rm C}} ds \\ = \rho\int_0^t (\boldsymbol{f}'(s), \boldsymbol{u}''_m(s))\,ds - \int_0^t (\boldsymbol{F}''(s), \boldsymbol{u}'_m(s))_{\Gamma_{\rm N}}\,ds + [(\boldsymbol{F}'(s), \boldsymbol{u}'_m(s))_{\Gamma_{\rm N}}]_{s=0}^{s=t} \\ + \int_0^t (F''_n(s), [\![u'_{mn}(s)]\!])_{\Gamma_{\rm C}}\,ds - [(F'_n(s), [\![u'_{mn}(s)]\!])_{\Gamma_{\rm C}}]_{s=0}^{s=t}$$

$$+\int_0^t (g''(s), \varphi_\epsilon([\![\boldsymbol{u}'_{m\tau}(s)]\!]))_{\Gamma_C}\,ds - \big[\big(g'(s), \varphi_\epsilon([\![\boldsymbol{u}'_{m\tau}(s)]\!])\big)_{\Gamma_C}\big]_{s=0}^{s=t}$$

for all $t \in [0,T]$, whence

$$\begin{aligned}
&\|\boldsymbol{u}''_m(t)\|_H^2 + \|\boldsymbol{u}'_m(t)\|_V^2 \\
&\qquad \le C\big(\|\boldsymbol{u}''_m(0)\|_H^2 + \|\dot{\boldsymbol{u}}_0\|_V^2 + \|\boldsymbol{f}\|^2_{H^1(0,T;H))} + \|\boldsymbol{F}\|^2_{H^2(0,T;L^2(\Gamma_N))} \\
&+ \|F_n\|^2_{H^2(0,T;L^2(\Gamma_C))} + \|g\|^2_{H^2(0,T;L^2(\Gamma_C))} + \epsilon^2\big) + C\int_0^t \big(\|\boldsymbol{u}''_m(s)\|_H^2 + \|\boldsymbol{u}'_m(s)\|_V^2\big)\,ds,
\end{aligned}$$

where we used $|\varphi_\epsilon(\boldsymbol{x})|^2 = |\boldsymbol{x}|^2 + \epsilon^2$. Then Gronwall's inequality implies that

$$\begin{aligned}
\|\boldsymbol{u}''_m(t)\|_H^2 + \|\boldsymbol{u}'_m(t)\|_V^2 &\le Ce^{Ct}(\|\boldsymbol{u}''_m(0)\|_H^2 + \|\dot{\boldsymbol{u}}_0\|_V^2 + \|\boldsymbol{f}\|^2_{H^1(0,T;H)} \\
&\quad + \|\boldsymbol{F}\|^2_{H^2(0,T;L^2(\Gamma_N))} + \|F_n\|^2_{H^2(0,T;L^2(\Gamma_C))} + \|g\|^2_{H^2(0,T;L^2(\Gamma_C))} + \epsilon^2)
\end{aligned}$$

for all $t \in [0,T]$.

A priori estimate 3: $\boldsymbol{u}''_m(0) \in H$. Multiplying (15) by $c''_k(t)$, taking the summation for $k = 1,\ldots,m$, and setting $t = 0$, we get

$$\begin{aligned}
&\rho\|\boldsymbol{u}''_m(0)\|_H^2 + a(\boldsymbol{u}_0, \boldsymbol{u}''_m(0)) + (g(0)\boldsymbol{\alpha}_\epsilon([\![\dot{\boldsymbol{u}}_{0\tau}]\!]), [\![\boldsymbol{u}''_{m\tau}(0)]\!])_{\Gamma_C} \\
&\qquad = \rho(\boldsymbol{f}(0), \boldsymbol{u}''_m(0)) + (\boldsymbol{F}(0), \boldsymbol{u}''_m(0))_{\Gamma_N} - (F_n(0), [\![u''_{mn}(0)]\!])_{\Gamma_C}.
\end{aligned}$$

It follows from integration by parts that

$$\begin{aligned}
\rho\|\boldsymbol{u}''_m(0)\|_H^2 &= -a(\boldsymbol{u}_0, \boldsymbol{u}''_m(0)) + \rho(\boldsymbol{f}(0), \boldsymbol{u}''_m(0)) + (\boldsymbol{F}(0), \boldsymbol{u}''_m(0))_{\Gamma_N} \\
&\quad - (F_n(0), [\![u''_{mn}(0)]\!])_{\Gamma_C} - (g(0)\boldsymbol{\alpha}_\epsilon([\![\dot{\boldsymbol{u}}_{0\tau}]\!]), [\![\boldsymbol{u}''_{m\tau}(0)]\!])_{\Gamma_C} \\
&= (\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{u}_0) + \rho\boldsymbol{f}(0), \boldsymbol{u}''_m(0)) - (\boldsymbol{\sigma}(\boldsymbol{u}_0)\boldsymbol{n} - \boldsymbol{F}(0), \boldsymbol{u}''_m(0))_{\Gamma_N} \\
&\quad + (\sigma_n(\boldsymbol{u}_0) - F_n(0), [\![u''_{mn}(0)]\!])_{\Gamma_C} + (\boldsymbol{\sigma}_\tau(\boldsymbol{u}_0) - g(0)\boldsymbol{\alpha}_\epsilon([\![\dot{\boldsymbol{u}}_{0\tau}]\!]), [\![\boldsymbol{u}''_{m\tau}(0)]\!])_{\Gamma_C} \\
&= (\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{u}_0) + \rho\boldsymbol{f}(0), \boldsymbol{u}''_m(0)),
\end{aligned}$$

where the compatibility conditions (10) were applied, which implies that

$$\|\boldsymbol{u}''_m(0)\|_H \le C(\|\operatorname{div}\boldsymbol{\sigma}(\boldsymbol{u}_0)\|_H + \|\boldsymbol{f}(0)\|_H).$$

3.6. Passage to limit as $m \to \infty$. Based on the *a priori* estimates presented above, a subsequence of $\{\boldsymbol{u}_m\}$ can be extracted, denoted by the same symbol for simplicity, and there exists some $\boldsymbol{u}_\epsilon \in W^{2,\infty}(0,T;H) \cap W^{1,\infty}(0,T;V)$ such that

$$\begin{aligned}
\boldsymbol{u}_m &\rightharpoonup \boldsymbol{u}_\epsilon \quad \text{weakly-}* \text{ in } L^\infty(0,T;V), \\
\boldsymbol{u}'_m &\rightharpoonup \boldsymbol{u}'_\epsilon \quad \text{weakly-}* \text{ in } L^\infty(0,T;V), \\
\boldsymbol{u}''_m &\rightharpoonup \boldsymbol{u}''_\epsilon \quad \text{weakly-}* \text{ in } L^\infty(0,T;H)
\end{aligned}$$

as $m \to \infty$. Moreover, $\boldsymbol{u}_\epsilon$ is bounded in $W^{2,\infty}(0,T;H) \cap W^{1,\infty}(0,T;V)$ uniformly with respect to $\epsilon \in (0,1]$.

Let $\boldsymbol{v} \in V_m$ and $\psi \in C_0^\infty(0,T)$ be arbitrary. From (15), we have

$$\int_0^T \psi(t)\big(\rho(\boldsymbol{u}_m''(t),\boldsymbol{v}) + a(\boldsymbol{u}_m(t),\boldsymbol{v}) + (g(t)\boldsymbol{\alpha}_\epsilon([\![\boldsymbol{u}_{m\tau}'(t)]\!]),[\![\boldsymbol{v}_\tau]\!])_{\Gamma_\mathrm{C}} - \langle \boldsymbol{L}(t),\boldsymbol{v}\rangle\big)\,dt = 0. \tag{17}$$

Here, it is to be noted that

$$W^{1,\infty}(0,T;H)\cap L^\infty(0,T;V) \subset H^1(0,T;H)\cap L^2(0,T;V) \hookrightarrow L^2(0,T;H)$$

and that the last embedding is compact (see [16, Theorem 5 on p. 84]); hence,

$$\boldsymbol{u}_m' \to \boldsymbol{u}_\epsilon' \quad \text{strongly in} \quad L^2(0,T;H) \quad \text{as} \quad m\to\infty.$$

This together with trace inequality

$$\begin{aligned}
\|[\![\boldsymbol{v}_\tau]\!]\|_{L^2(\Gamma_\mathrm{C})} &\le \|\boldsymbol{v}^+\|_{L^2(\Gamma_\mathrm{C})} + \|\boldsymbol{v}^-\|_{L^2(\Gamma_\mathrm{C})} \\
&\le C\big(\|\boldsymbol{v}^+\|_{L^2(\Omega^+)}^{1/2}\|\boldsymbol{v}^+\|_{H^1(\Omega^+)}^{1/2} + \|\boldsymbol{v}^-\|_{L^2(\Omega^-)}^{1/2}\|\boldsymbol{v}^-\|_{H^1(\Omega^-)}^{1/2}\big) \\
&\le C\|\boldsymbol{v}\|_H^{1/2}\|\boldsymbol{v}\|_V^{1/2},
\end{aligned}$$

implies that $[\![\boldsymbol{u}_{m\tau}']\!] \to [\![\boldsymbol{u}_{\epsilon\tau}']\!]$ strongly in $L^2(0,T;L^2(\Gamma_c)) = L^2((0,T)\times\Gamma_\mathrm{C})$ as $m\to\infty$. In particular, $[\![\boldsymbol{u}_{m\tau}']\!] \to [\![\boldsymbol{u}_{\epsilon\tau}']\!]$ a.e. in $(0,T)\times\Gamma_\mathrm{C}$.

Passing $m\to\infty$ in (17), it follows from the dominated convergence theorem that

$$\int_0^T \psi(t)(\rho(\boldsymbol{u}_\epsilon''(t),\boldsymbol{v}) + a(\boldsymbol{u}_\epsilon(t),\boldsymbol{v}) + (g(t)\boldsymbol{\alpha}_\epsilon([\![\boldsymbol{u}_{\epsilon\tau}'(t)]\!]),[\![\boldsymbol{v}_\tau]\!])_{\Gamma_\mathrm{C}} - \langle \boldsymbol{L}(t),\boldsymbol{v}\rangle)\,dt = 0,$$

which holds for all $\boldsymbol{v} \in V_m$ and, therefore, for all $\boldsymbol{v} \in V$, as a consequence of the density relation (17). Since $\psi \in C_0^\infty(0,T)$ is arbitrary, this proves the existence of a solution $\boldsymbol{u}_\epsilon$ for (13) and for (12).

3.7. Passage to limit as $\epsilon \to 0$. As previously mentioned, there exist a subsequence of $\{\boldsymbol{u}_\epsilon\}$, denoted by the same symbol for simplicity, and some $\boldsymbol{u} \in W^{2,\infty}(0,T;H)\cap W^{1,\infty}(0,T;V)$ such that

$$\begin{aligned}
\boldsymbol{u}_\epsilon &\rightharpoonup \boldsymbol{u} \quad \text{weakly-}* \text{ in} \quad L^\infty(0,T;V),\\
\boldsymbol{u}_\epsilon' &\rightharpoonup \boldsymbol{u}' \quad \text{weakly-}* \text{ in} \quad L^\infty(0,T;V),\\
\boldsymbol{u}_\epsilon'' &\rightharpoonup \boldsymbol{u}'' \quad \text{weakly-}* \text{ in} \quad L^\infty(0,T;H),
\end{aligned}$$

as $\epsilon \to 0$.

For an arbitrary $\tilde{\boldsymbol{v}} \in L^2(0,T;V)$, let us prove that

$$\begin{aligned}
\int_0^T &(\rho(\boldsymbol{u}''(t),\tilde{\boldsymbol{v}}(t)-\boldsymbol{u}'(t)) + a(\boldsymbol{u}(t),\tilde{\boldsymbol{v}}(t)-\boldsymbol{u}'(t)) \\
&+ (g(t),|[\![\tilde{\boldsymbol{v}}_\tau(t)]\!]|)_{\Gamma_\mathrm{C}} - (g(t),|[\![\boldsymbol{u}_\tau'(t)]\!]|)_{\Gamma_\mathrm{C}} - \langle \boldsymbol{L}(t),\tilde{\boldsymbol{v}}(t)-\boldsymbol{u}'(t)\rangle)\,dt \ge 0.
\end{aligned} \tag{18}$$

It follows from (12) that

$$\int_0^T (\rho(\boldsymbol{u}_\epsilon''(t), \tilde{\boldsymbol{v}}(t) - \boldsymbol{u}_\epsilon'(t)) + a(\boldsymbol{u}_\epsilon(t), \tilde{\boldsymbol{v}}(t) - \boldsymbol{u}_\epsilon'(t)) \\ + (g(t), \varphi_\epsilon([\![\tilde{\boldsymbol{v}}_\tau(t)]\!]))_{\Gamma_{\mathrm{C}}} - (g(t), \varphi_\epsilon([\![\boldsymbol{u}_{\epsilon\tau}'(t)]\!]))_{\Gamma_{\mathrm{C}}} - \langle \boldsymbol{L}(t), \tilde{\boldsymbol{v}}(t) - \boldsymbol{u}_\epsilon'(t)\rangle)\, dt \geq 0,$$

which is equivalent to

$$\int_0^T \big(\rho(\boldsymbol{u}_\epsilon''(t), \tilde{\boldsymbol{v}}(t)) + a(\boldsymbol{u}_\epsilon(t), \tilde{\boldsymbol{v}}(t)) + (g(t), \varphi_\epsilon([\![\tilde{\boldsymbol{v}}_\tau(t)]\!]))_{\Gamma_{\mathrm{C}}} - \langle \boldsymbol{L}(t), \tilde{\boldsymbol{v}}(t)\rangle\big)\, dt \\ \geq \frac{1}{2}\big(\rho\|\boldsymbol{u}_\epsilon'(T)\|_H^2 + a(\boldsymbol{u}_\epsilon(T), \boldsymbol{u}_\epsilon(T)) - \rho\|\dot{\boldsymbol{u}}_0\|_H^2 - a(\boldsymbol{u}_0, \boldsymbol{u}_0)\big) \\ + \int_0^T \big((g(t), \varphi_\epsilon([\![\boldsymbol{u}_{\epsilon\tau}'(t)]\!]))_{\Gamma_{\mathrm{C}}} - \langle \boldsymbol{L}(t), \boldsymbol{u}_\epsilon'(t)\rangle\big)\, dt. \quad (19)$$

According to [16, Theorem 5, p. 84], the embedding

$$W^{1,\infty}(0,T;H) \cap L^\infty(0,T;V) \hookrightarrow C([0,T];H)$$

is compact, so that

$$\boldsymbol{u}_\epsilon(0) \to \boldsymbol{u}(0) \quad \text{strongly in } H,$$
$$\boldsymbol{u}_\epsilon'(0) \to \boldsymbol{u}'(0), \boldsymbol{u}_\epsilon'(T) \to \boldsymbol{u}'(T) \quad \text{strongly in } H,$$
$$[\![\boldsymbol{u}_{\epsilon\tau}']\!] \to [\![\boldsymbol{u}_\tau']\!] \quad \text{strongly in } C([0,T]; L^2(\Gamma_{\mathrm{C}})).$$

Moreover, it holds

$$a(\boldsymbol{u}(T), \boldsymbol{u}(T)) \leq \liminf_{\epsilon\to 0} a(\boldsymbol{u}_\epsilon(T), \boldsymbol{u}_\epsilon(T)).$$

This is obtained by $a(\boldsymbol{u}_\epsilon(T), \boldsymbol{w}) \to a(\boldsymbol{u}(T), \boldsymbol{w})$ $(\epsilon \to 0)$ for any $\boldsymbol{w} \in V$, which results from

$$a(\boldsymbol{u}(T) - \boldsymbol{u}_\epsilon(T), \boldsymbol{w}) = \int_0^T \big(a(\boldsymbol{u}'(t) - \boldsymbol{u}_\epsilon'(t), \psi(t)\boldsymbol{w}) + a(\boldsymbol{u}(t) - \boldsymbol{u}_\epsilon(t), \psi'(t)\boldsymbol{w})\big)\, dt \to 0,$$

where $\psi \in C^\infty([0,\infty])$ is selected such that $\psi(0) = 0$ and $\psi(T) = 1$. Therefore, taking $\epsilon \to 0$ in (19) and noting that $\varphi_\epsilon([\![\boldsymbol{u}_{\epsilon\tau}']\!]) \to |[\![\boldsymbol{u}_\tau']\!]|$ in $C([0,T]; L^2(\Gamma_{\mathrm{C}}))$, we deduce the following:

$$\int_0^T (\rho(\boldsymbol{u}''(t), \tilde{\boldsymbol{v}}(t)) + a(\boldsymbol{u}(t), \tilde{\boldsymbol{v}}(t)) + (g(t), |[\![\tilde{\boldsymbol{v}}_\tau(t)]\!]|)_{\Gamma_{\mathrm{C}}} - \langle \boldsymbol{L}(t), \tilde{\boldsymbol{v}}(t)\rangle)\, dt \\ \geq \frac{1}{2}\big(\rho\|\boldsymbol{u}'(T)\|_H^2 + a(\boldsymbol{u}(T), \boldsymbol{u}(T)) - \rho\|\dot{\boldsymbol{u}}_0\|_H^2 - a(\boldsymbol{u}_0, \boldsymbol{u}_0)\big) \\ + \int_0^T ((g(t), |[\![\boldsymbol{u}_\tau'(t)]\!]|)_{\Gamma_{\mathrm{C}}} - \langle \boldsymbol{L}(t), \boldsymbol{u}'(t)\rangle)\, dt,$$

which is the desired inequality (18).

Now, we see that (18) implies (8), exploiting a technique based on the Lebesgue differentiation theorem (see [3, pp. 57, 58]). Consequently, we conclude the existence of a solution for problem (E').

3.8. Uniqueness. Let $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ be two solutions for (8). Then, setting $\boldsymbol{v} = \boldsymbol{u}_2'(t)$ ($\boldsymbol{v} = \boldsymbol{u}_1'(t)$, respectively) in the variational inequality that $\boldsymbol{u}_1$ ($\boldsymbol{u}_2$, respectively) satisfies and adding the resulting inequalities, we have

$$\frac{1}{2}\frac{d}{dt}(\rho\|\boldsymbol{u}_1'(t) - \boldsymbol{u}_2'(t)\|_H^2 + a(\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t), \boldsymbol{u}_1(t) - \boldsymbol{u}_2(t))) \leq 0,$$

immediately yielding $\boldsymbol{u}_1 = \boldsymbol{u}_2$. This completes the proof of the theorem.

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