

## ON THE JACOBIAN GROUP OF A CONE OVER A CIRCULANT GRAPH

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**Abstract.** For any given graph  $G$ , consider the graph  $\widehat{G}$  which is a cone over  $G$ . We study two important invariants of such a cone, namely, the complexity (the number of spanning trees) and the Jacobian of the graph. We prove that complexity of graph  $\widehat{G}$  coincides with the number of rooted spanning forests in  $G$  and the Jacobian of  $\widehat{G}$  is isomorphic to the cokernel of the operator  $I + L(G)$ , where  $L(G)$  is the Laplacian of  $G$  and  $I$  is the identity matrix. As a consequence, one can calculate the complexity of  $\widehat{G}$  as  $\det(I + L(G))$ .

As an application, we establish general structural theorems for the Jacobian of  $\widehat{G}$  in the case when  $G$  is a circulant graph or cobordism of two circulant graphs.

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**Keywords:** spanning tree, spanning forest, circulant graph, Laplacian matrix, cone over graph, Chebyshev polynomial.

### 1. Introduction

A *spanning tree* in a finite connected graph  $G$  is defined as a subgraph of  $G$  that contains all vertices of  $G$  and has no cycles. The number of spanning trees of the graph  $G$  is also called a *complexity* of  $G$ . It is a very important graph invariant and along with the pioneers in this field [1], formulas were found for some special graphs such as the wheel [2], fan [3], ladder [4], Möbius ladder [5], lattice [6], prism [7] and anti-prism [8]. However, one of the most significant and general results is the Kirchhoff matrix-tree theorem [9] which states that the complexity of  $G$  can be expressed as the product of nonzero Laplacian eigenvalues of  $G$ , divided by the number of its vertices. In this paper, we will also apply the idea [2] of using Chebyshev polynomials for counting various invariants of graphs arose.

Also, no less interesting invariant of a graph is the number of rooted spanning forests in a graph  $G$ . According to the classical result [10], this value can be found as determinant  $\det(I + L(G))$ . Here,  $L(G)$  is the *Laplacian matrix* of graph  $G$ . However, not many explicit formulas are known. One of the first results was obtained by O. Knill [11], who found the analytical formula of the number of rooted spanning forests in the complete graph  $K_n$ . Some formulas were obtained for bipartite graphs [12], cyclic, star, line graphs [11] and some others [13]. In our previous paper [14],

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we developed a new method for counting rooted spanning forests in circulant graphs. As for the number of unrooted forests, it has a much more complicated structure [15–17].

Another well-known invariant of a finite graph is *Jacobian group* (also known as the Picard group, critical group, sandpile group, dollar group). This concept was introduced independently by several authors [18–22], and [23]. In particular, the order of the Jacobian group coincides with the number of spanning trees for a graph. This fact is one of the reasons why interest in the Jacobian of a graph is manifested. However, the description of the Jacobian structure remains a difficult task and its structure is known only in several cases [18, 20, 24–27] and [28]. In this paper we intend to use the result [29] about the cokernel structure of Laplacian operator.

The paper is organized as follows.

Section 2 contains basic definitions and some known results on circulant graphs and circulant matrices. In Section 3, we describe general properties of cokernels for  $\mathbb{Z}$ -linear operators represented by circulant matrices. The main result of Section 4 is Theorem 1 which asserts that the number of spanning trees in the cone over a graph  $G$  coincides with the number of rooted spanning forests in  $G$ . In Section 5, we introduce a notion of the forest group for a graph  $G$  defined as the cokernel of  $\mathbb{Z}$ -linear operator  $I + L(G)$ . Then the main result of the section (Theorem 2) states that Jacobian of the cone over a graph  $G$  is isomorphic to the forest group of  $G$ . Section 6 is devoted to description of Jacobian groups for the cone over a circulant graph and the cone over cobordism of two circulant graphs. Lastly, in Section 7, we use the obtained results to calculate Jacobian group and number of spanning trees for cones over some simple families of graphs.

## 2. Basic definitions and preliminary facts

Let  $G$  be a finite graph without loops. We denote the vertex and edge set of  $G$  by  $V(G)$  and  $E(G)$ , respectively. Given  $u, v \in V(G)$ , we set  $a_{uv}$  to be equal to the number of edges between vertices  $u$  and  $v$ . The matrix  $A = A(G) = \{a_{uv}\}_{u, v \in V(G)}$  is called *the adjacency matrix* of the graph  $G$ . The degree  $d(v)$  of a vertex  $v \in V(G)$  is defined by  $d(v) = \sum_{u \in V(G)} a_{uv}$ . Let  $D = D(G)$  be the diagonal matrix of the size  $|V(G)|$  with  $d_{vv} = d(v)$ . The matrix  $L = L(G) = D(G) - A(G)$  is called *the Laplacian matrix*, or simply *Laplacian*, of the graph  $G$ .

Consider the Laplacian  $L(G)$  as a homomorphism  $\mathbb{Z}^n \rightarrow \mathbb{Z}^n$ , where  $n$  is the number of vertices in  $G$ . The cokernel  $\text{coker}(L(G)) = \mathbb{Z}^n / \text{im}(L(G))$  is an Abelian group. It can be uniquely represented in the form

$$\text{coker}(L(G)) \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_n},$$

where  $d_i$  satisfy the conditions  $d_i | d_{i+1}$  ( $1 \leq i \leq n$ ).

Suppose that the graph  $G$  is connected, then the groups  $\mathbb{Z}_{d_1}, \mathbb{Z}_{d_2}, \dots, \mathbb{Z}_{d_{n-1}}$  are finite, and  $\mathbb{Z}_{d_n} = \mathbb{Z}$ . In this case, we define Jacobian of the graph  $G$  as

$$\text{Jac}(G) \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_{n-1}}.$$

In other words,  $\text{Jac}(G)$  is isomorphic to the torsion subgroup of  $\text{coker}(L(G))$ .

Let  $s_1, s_2, \dots, s_k$  be integers such that  $1 \leq s_1 < s_2 < \dots < s_k \leq \frac{n}{2}$ . The graph  $C_n(s_1, s_2, \dots, s_k)$  with  $n$  vertices  $0, 1, 2, \dots, n-1$  is called *circulant graph* if the vertex  $i$ ,  $0 \leq i \leq n-1$  is adjacent to the vertices  $i \pm s_1, i \pm s_2, \dots, i \pm s_k \pmod{n}$ . When  $s_k < \frac{n}{2}$  all vertices of a graph have even degree  $2k$ . If  $n$  is even and  $s_k = \frac{n}{2}$ , then all vertices have odd degree  $2k-1$ . It is well known that the circulant  $C_n(s_1, s_2, \dots, s_k)$  is connected if and only if  $\text{gcd}(s_1, s_2, \dots, s_k, n) = 1$ .

We call an  $n \times n$  matrix *circulant*, and denote it by  $\text{circ}(a_0, a_1, \dots, a_{n-1})$  if it is of the form

$$\text{circ}(a_0, a_1, \dots, a_{n-1}) = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2} \\ & \vdots & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{pmatrix}.$$

It is easy to see that adjacency and Laplacian matrices of the circulant graph are circulant matrices. The converse is also true. If the Laplacian matrix of a graph is circulant then the graph is also circulant.

Recall [30] that the eigenvalues of matrix  $C = \text{circ}(a_0, a_1, \dots, a_{n-1})$  are given by the following simple formulas  $\lambda_j = P(\varepsilon_n^j)$ ,  $j = 0, 1, \dots, n-1$ , where  $P(x) = a_0 + a_1x + \dots + a_{n-1}x^{n-1}$  and  $\varepsilon_n$  is an order  $n$  primitive root of the unity. Moreover, the circulant matrix  $C = P(T)$ , where  $T_n = \text{circ}(0, 1, 0, \dots, 0)$  is the matrix representation of the shift operator  $T_n : (x_0, x_1, \dots, x_{n-2}, x_{n-1}) \rightarrow (x_1, x_2, \dots, x_{n-1}, x_0)$ .

### 3. Cokernels of linear operators

Let  $P(z)$  be a bimononic integer Laurent polynomial. That is  $P(z) = z^p + a_1z^{p+1} + \dots + a_{s-1}z^{p+s-1} + z^{p+s}$  for some integers  $p, a_1, a_2, \dots, a_{s-1}$  and some positive integer  $s$ . Introduce the following *companion matrix*  $\mathcal{A}$  for polynomial

$$P(z): \quad \mathcal{A} = \left( \begin{array}{c|c} \mathbf{O} & I_{s-1} \\ \hline -1, -a_1, \dots, -a_{s-1} & \end{array} \right),$$

where  $I_{s-1}$  is the identity  $(s-1) \times (s-1)$  matrix. We note that  $\mathcal{A}$  is invertible and inverse matrix  $\mathcal{A}^{-1}$  is also integer matrix.

Let  $\mathbb{A} = \langle \alpha_j, j \in \mathbb{Z} \rangle$  be a free Abelian group freely generated by elements  $\alpha_j$ ,  $j \in \mathbb{Z}$ . Each element of  $\mathbb{A}$  is a linear combination  $\sum_j c_j \alpha_j$  with integer coefficients  $c_j$ . Define the shift operator  $T : \mathbb{A} \rightarrow \mathbb{A}$  as a  $\mathbb{Z}$ -linear operator acting on generators of  $\mathbb{A}$  by the rule  $T : \alpha_j \rightarrow \alpha_{j+1}$ ,  $j \in \mathbb{Z}$ . Then  $T$  is an endomorphism of  $\mathbb{A}$ .

Let  $P(z)$  be an arbitrary Laurent polynomial with integer coefficients, then  $A = P(T)$  is also an endomorphism of  $\mathbb{A}$ . Since  $A$  is a linear combination of powers of  $T$ , the action of  $A$  on generators  $\alpha_j$  can be given by the infinite set of linear transformations  $A : \alpha_j \rightarrow \sum_i a_{i,j} \alpha_i$ ,  $j \in \mathbb{Z}$ . Here all sums under consideration are finite. We set  $\beta_j = \sum_i a_{i,j} \alpha_i$ . Then  $\text{im } A$  is a subgroup of  $\mathbb{A}$  generated by  $\beta_j$ ,  $j \in \mathbb{Z}$ . Hence,  $\text{coker } A = \mathbb{A} / \text{im } A$  is an abstract Abelian group  $\langle x_i, i \in \mathbb{Z} \mid \sum_i a_{i,j} x_i = 0, j \in \mathbb{Z} \rangle$ .

$\mathbb{Z}\rangle$  generated by  $x_i, i \in \mathbb{Z}$  with the set of defining relations  $\sum_i a_{i,j}x_i = 0, j \in \mathbb{Z}$ . Here  $x_j$  are images of  $\alpha_j$  under the canonical homomorphism  $\mathbb{A} \rightarrow \mathbb{A}/\text{im } A$ . Since  $T$  and  $A = P(T)$  commute, subgroup  $\text{im } A$  is invariant under the action of  $T$ . Hence, the actions of  $T$  and  $A$  are well defined on the factor group  $\mathbb{A}/\text{im } A$  and are given by  $T : x_j \rightarrow x_{j+1}$  and  $A : x_j \rightarrow \sum_i a_{i,j}x_i$  respectively.

This allows to present the group  $\mathbb{A}/\text{im } A$  as follows  $\langle x_i, i \in \mathbb{Z} \mid P(T)x_j = 0, j \in \mathbb{Z} \rangle$ . In a similar way, given a set  $P_1(z), P_2(z), \dots, P_s(z)$  of Laurent polynomials with integer coefficients, one can define the group  $\langle x_i, i \in \mathbb{Z} \mid P_1(T)x_j = 0, P_2(T)x_j = 0, \dots, P_s(T)x_j = 0, j \in \mathbb{Z} \rangle$ .

We will use the following proposition. By  $I = I_n$  we denote the identity matrix of order  $n$ .

**Proposition 1.** *Let  $P(z)$  be a bimonic Laurent polynomial with integer coefficients and  $\mathcal{A}$  be a companion matrix of  $P(z)$ . Consider  $L = P(T_n) : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$  as a  $\mathbb{Z}$ -linear operator. Then*

$$\text{coker } L \cong \text{coker}(\mathcal{A}^n - I).$$

PROOF. Since the Laurent polynomial  $P(z)$  is bimonic, it can be represented in the form

$$P(z) = z^p + a_1z^{p+1} + \dots + a_{s-1}z^{p+s-1} + z^{p+s},$$

where  $p, s, a_1, a_2, \dots, a_{s-1}$  are integers and  $s > 0$ . Then the corresponding companion matrix  $\mathcal{A}$  is

$$\left( \begin{array}{c|c} 0 & I_{s-1} \\ \hline -1, -a_1, \dots, -a_{s-1} & \end{array} \right).$$

Let  $T$  be the shift operator defined by  $T : x_j \rightarrow x_{j+1}, j \in \mathbb{Z}$ . Note that for any  $j \in \mathbb{Z}$  the relations  $P(T)x_j = 0$  can be written as  $x_{j+s} = -x_j - a_1x_{j+1} - \dots - a_{s-1}x_{j+s-1}$ .

Let  $\mathbf{x}_j = (x_{j+1}, x_{j+2}, \dots, x_{j+s})^t$  be  $s$ -tuple of generators  $x_{j+1}, x_{j+2}, \dots, x_{j+s}$ . Then the relation  $P(T)x_j = 0$  is equivalent to  $\mathbf{x}_j = \mathcal{A}\mathbf{x}_{j-1}$ . Hence, we have  $\mathbf{x}_1 = \mathcal{A}\mathbf{x}_0$  and  $\mathbf{x}_{-1} = \mathcal{A}^{-1}\mathbf{x}_0$ , where  $\mathbf{x}_0 = (x_1, x_2, \dots, x_s)^t$ . So,  $\mathbf{x}_j = \mathcal{A}^j \mathbf{x}_0$  for any  $j \in \mathbb{Z}$ . Conversely, the latter implies  $\mathbf{x}_j = \mathcal{A}\mathbf{x}_{j-1}$  and, as a consequence,  $P(T)x_j = 0$  for all  $j \in \mathbb{Z}$ .

Consider  $\text{coker } A = \mathbb{A}/\text{im } A$  as an abstract Abelian group with the following representation  $\langle x_i, i \in \mathbb{Z} \mid P(T)x_j = 0, j \in \mathbb{Z} \rangle$ . We show that  $\text{coker } A \cong \mathbb{Z}^s$ . Indeed,

$$\begin{aligned} \text{coker } A &= \langle x_i, i \in \mathbb{Z} \mid P(T)x_j = 0, j \in \mathbb{Z} \rangle \\ &= \langle x_i, i \in \mathbb{Z} \mid x_j + a_1x_{j+1} + \dots + a_{s-1}x_{j+s-1} + x_{j+s} = 0, j \in \mathbb{Z} \rangle \\ &= \langle x_i, i \in \mathbb{Z} \mid (x_{j+1}, x_{j+2}, \dots, x_{j+s})^t = \mathcal{A}(x_j, x_{j+1}, \dots, x_{j+s-1})^t, j \in \mathbb{Z} \rangle \\ &= \langle x_i, i \in \mathbb{Z} \mid (x_{j+1}, x_{j+2}, \dots, x_{j+s})^t = \mathcal{A}^j(x_1, x_2, \dots, x_s)^t, j \in \mathbb{Z} \rangle \\ &= \langle x_1, x_2, \dots, x_s \mid - \rangle \cong \mathbb{Z}^s. \end{aligned}$$

Now, our aim is to find cokernel of  $L = P(T_n)$ . In the operator notations

$$\text{coker } L = \langle x_i, i \in \mathbb{Z} \mid P(T)x_j = 0, (T^n - 1)x_j, j \in \mathbb{Z} \rangle.$$

We set  $B = T^n - 1$  and note [29, Lemma 3.1] that

$$\text{coker } L \cong \text{coker } A / \text{im}(B|_{\text{coker } A}) \cong \text{coker}(B|_{\text{coker } A}).$$

We describe the action of the endomorphism  $B|_{\text{coker } A}$  on the coker  $A$ . Since the operators  $A = P(T)$  and  $T$  commute, the action  $T|_{\text{coker } A} : x_j \rightarrow x_{j+1}$ ,  $j \in \mathbb{Z}$  on the coker  $A$  is well defined. First of all, we describe the action of  $T|_{\text{coker } A}$  on the set of generators  $x_1, x_2, \dots, x_s$ . For any  $i = 1, \dots, s-1$ , we have  $T|_{\text{coker } A}(x_i) = x_{i+1}$  and  $T|_{\text{coker } A}(x_s) = x_{s+1} = -x_1 - a_1x_2 - \dots - a_{s-2}x_{s-1} - a_{s-1}x_s$ . Hence, the action of  $T|_{\text{coker } A}$  on the coker  $A$  is given by the matrix  $\mathcal{A}$ . Considering  $\mathcal{A}$  as an endomorphism of the coker  $A$ , we can write  $T|_{\text{coker } A} = \mathcal{A}$ . Then,  $B|_{\text{coker } A} = \mathcal{A}^n - I$  and  $\text{coker } L \cong \text{coker}(\mathcal{A}^n - I)$ .  $\square$

#### 4. The number of spanning trees of a cone over a graph

The *joint* of graphs  $G_1$  and  $G_2$  is called the graph  $G = G_1 * G_2$ , of order  $m + n$ , obtained from the disjoint union of  $G_1$  of order  $m$ , and  $G_2$  of order  $n$ , by additionally joining every vertex of  $G_1$  to every vertex of  $G_2$ . If  $G_2 = K_1$  (the one-vertex graph with no edges) we are going to call the graph  $G = G_1 * K_1$  a *cone over graph*  $G_1$ .

Let  $G$  be a graph on  $n$  vertices. We define  $\chi_G(\lambda) = \det(\lambda I_n - L(G))$  as the characteristic polynomial of matrix  $L(G)$ , which is Laplaican matrix of graph  $G$ . Its extended form is

$$\chi_G(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda.$$

The theorem by Kelmans and Chelnokov [10] states that the absolute value of coefficient  $c_k$  of  $\chi_G(\lambda)$  coincides with the number of rooted spanning  $k$ -forests in the graph  $G$ . So, the number of rooted spanning forests of the graph  $G$  can be found by the formula

$$\begin{aligned} f(G) &= f_1 + f_2 + \dots + f_n = |c_1 - c_2 + c_3 - \dots + (-1)^{n-1}| \\ &= (-1)^n \chi_G(-1) = \det(I_n + L(G)). \end{aligned} \quad (1)$$

This result was independently obtained by many authors: see, for example, [31, 11], and [14].

The main result of this section is the following theorem.

**Theorem 1.** *The number of spanning trees  $\tau(\widehat{G})$  in the graph  $\widehat{G}$ , which is a cone over a graph  $G$ , coincides with the number of rooted spanning forests  $f(G)$  in the graph  $G$ .*

PROOF. As a corollary of the well-known Matrix-Tree-Theorem [9], the number of spanning trees of graph  $G$  of order  $n$ , can be found by the next formula  $\tau(G) = \frac{(-1)^{n-1}}{n} \chi'_G(0)$ . According to the result by Kelmans [32, 33] (see also [34, Corollary 3.7]), for characteristic polynomial of a joint of two graphs  $G_1$  and  $G_2$ , of order  $m$  and  $n$ , we have

$$\chi_{G_1 * G_2}(x) = \frac{x(x - n - m)}{(x - n)(x - m)} \chi_{G_1}(x - n) \chi_{G_2}(x - m).$$

As a consequence, for a graph  $\widehat{G} = G_1 * G_2$ , where  $G_1 = G$  and  $G_2 = K_1$ , we obtain

$$\begin{aligned} \tau(\widehat{G}) &= \frac{(-1)^n}{n+1} \chi'_{\widehat{G}}(0) \\ &= \frac{(-1)^n}{n+1} \chi'_{G * K_1}(0) = \frac{(-1)^m}{m+1} \left( \frac{x(x-1-n)}{(x-1)(x-n)} \chi_{K_1}(x-n) \chi_G(x-1) \right)'_{x=0}. \end{aligned}$$

It is known that  $\chi_{K_1}(x) = x$ , so we obtain

$$\tau(\widehat{G}) = \frac{(-1)^n}{n+1} \lim_{x \rightarrow 0} \frac{(x-1-n)(x-n) \chi_G(x-1)}{(x-1)(x-n)} = (-1)^n \chi_G(-1).$$

By making use of formula (1) we finish the proof.  $\square$

The following corollary gives a convenient way to calculate the complexity of cone over a graph.

**Corollary 1.** *The number of spanning trees in the cone over a graph  $G$  is given by the formula  $|\chi_G(-1)|$ , where  $\chi_G(x)$  is the Laplacian characteristic polynomial of  $G$ .*

REMARK TO THEOREM 1. There is a natural way to get a one-to-one correspondence between spanning trees in the cone  $\widehat{G}$  and rooted spanning forests in the graph  $G$ .

Indeed, consider  $\widehat{G}$  as a joint  $G * \{v_0\}$  of  $G$  with one-vertex graph  $\{v_0\}$ . Let  $t$  be a spanning tree in  $\widehat{G}$ . We note that  $v_0$  is a vertex of  $t$ . Let  $v_0 v_j$ ,  $j = 1, 2, \dots, k$  be all the edges of graph  $t$  coming from vertex  $v_0$ . Then  $f = t \cap G$  is a spanning forest in  $G$  consisting of  $k$  trees  $t_1, t_2, \dots, t_k$  chosen in such a way that  $v_j$  is a vertex of  $t_j$ . So, the pairs  $(t_j, v_j)$ ,  $j = 1, 2, \dots, k$  form a rooted spanning forest in  $G$ . In turn, if  $(t_j, v_j)$ ,  $j = 1, 2, \dots, k$  is a rooted spanning forest in  $G$ , then the graph  $t$  obtained as a union of edges  $v_0 v_j$  and trees  $t_j$ ,  $j = 1, 2, \dots, k$  is a spanning tree in  $\widehat{G}$ .

### 5. The Jacobian of a cone over a graph and the forest group

The aim of the current section is to prove the following theorem.

**Theorem 2.** *Let  $G$  be a graph on  $n$  vertices. Then Jacobian of the cone over  $G$  is isomorphic to the cokernel of linear operator  $I_n + L(G)$ . Here  $L(G)$  is the Laplacian matrix of  $G$  and  $I_n$  is the identity matrix of order  $n$ .*

PROOF. For any given graph  $G$  on  $n$  vertices denote by  $L(G)$  the Laplacian matrix of the graph  $G$  and by  $\widehat{G}$  a graph that is a cone over graph  $G$ . It easy to see that the Laplacian matrix of  $\widehat{G}$  can be represented in the following form  $L(\widehat{G}) = \begin{pmatrix} n & -\mathbf{1}_n \\ -\mathbf{1}_n^T & I_n + L(G) \end{pmatrix}$ , where  $I_n$  is an identity matrix of order  $n$  and  $\mathbf{1}_n$  is a vector  $(1, 1, \dots, 1)$  of length  $n$ . To find the Jacobian of the  $\widehat{G}$  we use the following useful relation between the structure of the Laplacian matrix and the Jacobian of a graph [24].

Consider the Laplacian  $L(\widehat{G})$  as a homomorphism  $\mathbb{Z}^{n+1} \rightarrow \mathbb{Z}^{n+1}$ , where  $n+1$  is the number of vertices in  $\widehat{G}$ . The cokernel  $\text{coker}(L(\widehat{G})) = \mathbb{Z}^{n+1}/\text{im}(L(\widehat{G}))$  is an Abelian group. Let

$$\text{coker}(L(\widehat{G})) \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_{n+1}}$$

be its Smith normal form satisfying the conditions  $d_i | d_{i+1}$ ,  $1 \leq i \leq n$ . As the graph  $\widehat{G}$  is connected, the groups  $\mathbb{Z}_{d_1}, \mathbb{Z}_{d_2}, \dots, \mathbb{Z}_{d_n}$  are finite, and  $\mathbb{Z}_{d_{n+1}} = \mathbb{Z}$ . Here  $d_i = \delta_i / \delta_{i-1}$ , where  $\delta_i$ ,  $i = 1, 2, \dots, n+1$ , is the greatest common divisor of all  $i \times i$  minors of matrix  $L(\widehat{G})$  and  $\delta_0 = 1$ . Then,

$$\text{Jac}(\widehat{G}) \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_n}$$

is the Jacobian of the graph  $\widehat{G}$ .

To calculate the Smith normal form of a given matrix one can use the following basic operations to convert the matrix to a diagonal form.

- 1°. Add arbitrary integer linear combination of rows to any other row.
- 2°. Add arbitrary integer linear combination of columns to any other column.
- 3°. Interchange any two rows or columns.

The matrix  $\begin{pmatrix} n & -\mathbf{1}_n \\ -\mathbf{1}_n^T & I_n + L(G) \end{pmatrix}$  is the Laplacian matrix for graph  $\widehat{G}$ . So, the sum of all rows and the sum of all columns in this matrix are zero vectors. Adding all the other rows to the first row we get zero first row. Then we add to the first column the all remained columns to get zero first column. As such, one can easily check that up to operations 1° – 3° of the above matrix is equivalent to matrix  $\begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0}^T & I_n + L(G) \end{pmatrix}$ , where  $\mathbf{0}$  is a zero vector of length  $n$ . Therefore, all nonzero elements of the Smith normal form are completely defined by the matrix  $I_n + L(G)$ .  $\square$

We note that matrix  $I_n + L(G)$  is always non-singular. Also,  $\text{coker}(I_n + L(G))$  is an Abelian group whose size  $\det(I_n + L(G))$  is equal to the number of rooted spanning forests in graph  $G$ . So, it is natural to call  $\text{coker}(I_n + L(G))$  as a *forest group* of  $G$  and denote it by  $F(G)$ . Then, then main statement of Theorem 2 can be rephrased as follows:

*The Jacobian of the cone over a graph  $G$  is isomorphic to its forest group  $F(G)$ .*

## 6. The Jacobian of a cone over a circulant graph

This section is devoted to Jacobians of specific classes of graphs. They are cones over three families of graphs. Namely, cones over circulant graphs  $G = C_n(s_1, s_2, \dots, s_k)$  with even valency, circulant graphs  $G = C_{2n}(s_1, s_2, \dots, s_k, n)$  with odd valency and cobordisms of two circulant graphs. The typical example of the graph in the first family is a cone over the cyclic graph  $C_n = C_n(1)$  also known as Wheel graph  $W(n)$ . The second family contains a cone over the Möbius ladder  $C_{2n}(1, n)$ . The third set is represented by a cone over the Prism graph  $Pr(n)$ .

Denote by  $\widehat{G}$  the cone over graph  $G$ . Using the results of Section 5, we establish general structural theorems for  $\text{Jac}(\widehat{G})$  or, equivalently, for the forest group  $F(G) = \text{coker}(I + L(G))$ .

**6.1. The forest group of a circulant graph of even valency.** Consider a  $2k$ -valent circulant graph  $G = C_n(s_1, s_2, \dots, s_k)$ , where  $1 \leq s_1 < \dots < s_k < \frac{n}{2}$ . Its Laplacian has the form

$$L(G) = 2kI_n - \sum_{l=1}^k (T_n^{s_l} + T_n^{-s_l}),$$

where  $T_n = \text{circ}(0, 1, 0, \dots, 0)$  is the  $(n \times n)$  circulant matrix representing the shift operator  $(x_1, x_2, \dots, x_{n-1}, x_n) \rightarrow (x_2, x_3, \dots, x_n, x_1)$ . Then the forest group  $\text{coker}(I + L(G))$  has the following presentation:

$$\left\langle x_i, i \in \mathbb{Z} \mid (2k+1)x_j - \sum_{l=1}^k (x_{j+s_l} + x_{j-s_l}) = 0, x_{j+n} = x_j, j \in \mathbb{Z} \right\rangle.$$

By Proposition 1, we conclude that  $\text{coker}(I + L(G))$  is isomorphic to the  $\text{coker}(\mathcal{A}^n - I)$ , where  $\mathcal{A}$  is a companion matrix of the Laurent polynomial  $2k+1 - \sum_{l=1}^k (z^{s_l} + z^{-s_l})$ . Combine this observation with Theorem 2, we get the following result.

**Theorem 3.** *Let  $\widehat{G}$  be a cone over the circulant graph  $G = C_n(s_1, s_2, \dots, s_k)$ , where  $1 \leq s_1 < s_2 < \dots < s_k < \frac{n}{2}$ . Then  $\text{Jac}(\widehat{G})$  is isomorphic to  $\text{coker}(\mathcal{A}^n - I)$ , where  $\mathcal{A}$  is a companion matrix of the Laurent polynomial  $2k+1 - \sum_{l=1}^k (z^{s_l} + z^{-s_l})$ .*

**6.2. Forest group of circulant graph of odd valency.** Consider a  $(2k+1)$ -valent circulant graph of the form

$$G = C_{2n}(s_1, s_2, \dots, s_k, n), \text{ where } 1 \leq s_1 < s_2 < \dots < s_k < n.$$

In this case, the Laplacian matrix of  $G$  is

$$(2k+1)I_{2n} - T_{2n}^n - \sum_{j=1}^k (T_{2n}^{s_j} + T_{2n}^{-s_j}),$$

where  $T_{2n} = \text{circ}(0, 1, 0, \dots, 0)$  is a  $(2n \times 2n)$  circulant matrix. In order to get the forest group of  $G$ , we have to find  $\text{coker}(I + L(G))$ . It can be viewed as an infinitely generated Abelian group satisfying the following set of relations:

$$\left\langle x_i, i \in \mathbb{Z} \mid (2k+2)x_j - x_{j+n} - \sum_{l=1}^k (x_{j+s_l} + x_{j-s_l}) = 0, x_{j+2n} = x_j, j \in \mathbb{Z} \right\rangle.$$

By making use of the shift operator  $T : x_j \rightarrow x_{j+1}, j \in \mathbb{Z}$ , we rewrite the last formula as

$$\left\langle x_i, i \in \mathbb{Z} \mid \left( 2k+2 - T^n - \sum_{l=1}^k (T^{s_l} + T^{-s_l}) \right) x_j = 0, (T^{2n} - 1)x_j = 0, j \in \mathbb{Z} \right\rangle.$$



We can increase the list of relations by ones that are linear combinations of elements of a given set. One of such combinations is

$$\begin{aligned} (T^{2n} - 1) + B(T) \left( 2k + 2 - T^n - \sum_{l=1}^k (T^{s_l} + T^{-s_l}) \right) \\ = \left( 2k + 2 - \sum_{l=1}^k (T^{s_l} + T^{-s_l}) \right)^2 - 1, \end{aligned}$$

where

$$B(T) = 2k + 2 + T^n - \sum_{l=1}^k (T^{s_l} + T^{-s_l}).$$

In turn,  $T^{2n} - 1$  is a linear combination of  $2k + 2 - T^n - \sum_{l=1}^k (T^{s_l} + T^{-s_l})$  and  $\left( 2k + 2 - \sum_{l=1}^k (T^{s_l} + T^{-s_l}) \right)^2 - 1$ . So, it can be replaced by the latter expression in the group presentation. Hence,  $\text{coker}(I + L(G))$  admits the following presentation

$$\left\langle x_i, i \in \mathbb{Z} \mid \left( \left( 2k + 2 - T^n - \sum_{l=1}^k (T^{s_l} + T^{-s_l}) \right) x_j = 0, \right. \right. \\ \left. \left. \left( 2k + 2 - \sum_{l=1}^k (T^{s_l} + T^{-s_l}) \right)^2 - 1 \right) x_j = 0, j \in \mathbb{Z} \right\rangle.$$

By Proposition 1, the forest group  $\text{coker}(I + L(G))$  is isomorphic to

$$\text{coker} \left( \mathcal{A}^n - (2k + 2)I + \sum_{l=1}^k (\mathcal{A}^{s_l} + \mathcal{A}^{-s_l}) \right),$$

where  $\mathcal{A}$  is a companion matrix of the Laurent polynomial

$$\left( 2k + 2 - \sum_{l=1}^k (z^{s_l} + z^{-s_l}) \right)^2 - 1.$$

Applying Theorem 2, we rewrite the obtained result in the following form.

**Theorem 4.** *Let  $\widehat{G}$  be a cone over the circulant graph*

$$G = C_{2n}(s_1, s_2, \dots, s_k, n), 1 \leq s_1 < s_2 < \dots < s_k < n.$$

*Then  $\text{Jac}(\widehat{G})$  is isomorphic to  $\text{coker} \left( \mathcal{A}^n - (2k + 2)I + \sum_{j=1}^k (\mathcal{A}^{s_j} + \mathcal{A}^{-s_j}) \right)$ , where  $\mathcal{A}$  is a companion matrix of the Laurent polynomial  $\left( 2k + 2 - \sum_{j=1}^k (z^{s_j} + z^{-s_j}) \right)^2 - 1$ .*

**6.3. The Jacobian of a cone over cobordism of two circulant graphs.**

Consider two circulant graphs on  $n$  vertices, namely  $C_1 = C_n(s_{1,1}, s_{1,2}, \dots, s_{1,k})$  and

$C_2 = C_n(s_{1,1}, s_{1,2}, \dots, s_{1,l})$ , with  $k$  and  $l$  jumps respectively. Then the cobordism of two circulant graphs  $C_1$  and  $C_2$  is a graph  $G$  which is obtained from  $C_1$  and  $C_2$  by connecting every  $i$ th vertex of  $C_1$  with the corresponding  $i$ th vertex of  $C_2$ . The Laplacian matrix of  $G$  has the form

$$\begin{pmatrix} (2k+1)I_n - \sum_{r=1}^k (T_n^{s_{1,r}} + T_n^{-s_{1,r}}) & -I_n \\ -I_n & (2l+1)I_n - \sum_{r=1}^l (T_n^{s_{2,r}} + T_n^{-s_{2,r}}) \end{pmatrix}.$$

The simplest example is the prism graph  $Pr(n)$  that is a cobordism of two cyclic graphs  $C_n(1)$ .

The complexity and other spectral properties of the cobordism graph  $G$  were investigated in [35].

Denote by  $\widehat{G}$  the cone over graph  $G$ . The aim of this subsection is to find cokernel of  $I + L(G)$ , that is  $\text{Jac}(\widehat{G})$ . To do this, we use two bi-infinite sequences  $x_j, y_j, j \in \mathbb{Z}$ . Then the cokernel of the linear operator  $I + L(G)$  is isomorphic to the group

$$\left\langle x_i, y_i, i \in \mathbb{Z} \mid (2k+2)x_j - \sum_{r=1}^k (x_{j+s_{1,r}} + x_{j-s_{1,r}}) - y_j = 0, x_{j+n} - x_j = 0, \right. \\ \left. (2l+2)y_j - \sum_{r=1}^l (y_{j+s_{2,r}} + y_{j-s_{2,r}}) - x_j = 0, y_{j+n} - y_j = 0, j \in \mathbb{Z} \right\rangle.$$

We note that

$$y_j = (2k+2)x_j - \sum_{r=1}^k (x_{j+s_{1,r}} + x_{j-s_{1,r}})$$

is an integer linear combinations of  $x_j, j \in \mathbb{Z}$ . Equivalently, in the operator form,

$$y_j = \left( 2k+2 - \sum_{r=1}^k (T^{s_{1,r}} + T^{-s_{1,r}}) \right) x_j.$$

Then the group above is isomorphic to

$$\left\langle x_i \mid \left( \left( 2k+2 - \sum_{r=1}^k (T^{s_{1,r}} + T^{-s_{1,r}}) \right) \left( 2l+2 - \sum_{r=1}^l (T^{s_{2,r}} + T^{-s_{2,r}}) \right) - 1 \right) x_j = 0, \right. \\ \left. (T^n - 1)x_j, j \in \mathbb{Z} \right\rangle.$$

By Proposition 1 and Theorem 2, we get the following result.

**Theorem 5.** *Let  $\widehat{G}$  be a cone over the cobordism graph  $G$ . Then the Jacobian  $\text{Jac}(\widehat{G})$  is isomorphic to the cokernel of the linear operator  $\mathcal{A}^n - I$ , where  $\mathcal{A}$  is a companion matrix of the Laurent polynomial*

$$\left( 2k+2 - \sum_{r=1}^k (z^{s_{1,r}} + z^{-s_{1,r}}) \right) \left( 2l+2 - \sum_{r=1}^l (z^{s_{2,r}} + z^{-s_{2,r}}) \right) - 1.$$

Table 1

$n$	$\text{Jac}(\widehat{M}(n))$	$ \text{Jac}(\widehat{M}(n)) $
3	$\mathbb{Z}_4^3 \oplus \mathbb{Z}_{28}$	1792
4	$\mathbb{Z}_{69} \oplus \mathbb{Z}_{345}$	23805
5	$\mathbb{Z}_{209} \oplus \mathbb{Z}_{1463}$	305767
6	$\mathbb{Z}_5 \oplus \mathbb{Z}_{880}^2$	3872000
7	$\mathbb{Z}_{2639} \oplus \mathbb{Z}_{18473}$	48750247
8	$\mathbb{Z}_{11067} \oplus \mathbb{Z}_{55335}$	612392445
9	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_{8284} \oplus \mathbb{Z}_{57988}$	7685961472
10	$\mathbb{Z}_5 \oplus \mathbb{Z}_{25} \oplus \mathbb{Z}_{27775}^2$	96431328125
11	$\mathbb{Z}_{415711} \oplus \mathbb{Z}_{2909977}$	1209709448647
12	$\mathbb{Z}_{1742112} \oplus \mathbb{Z}_{8710560}$	15174771102720
13	$\mathbb{Z}_{5214689} \oplus \mathbb{Z}_{36502823}$	190350869567047
14	$\mathbb{Z}_5 \oplus \mathbb{Z}_{21852805}^2$	2387725431840125
15	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_{16352996} \oplus \mathbb{Z}_{114470972}$	29951093555713792
16	$\mathbb{Z}_{274116549} \oplus \mathbb{Z}_{1370582745}$	375699412178347005
17	$\mathbb{Z}_{820512241} \oplus \mathbb{Z}_{5743585687}$	4712682363415894567
18	$\mathbb{Z}_5 \oplus \mathbb{Z}_{3438451280}^2$	59114736024668192000
19	$\mathbb{Z}_{10292304751} \oplus \mathbb{Z}_{72046133257}$	741520759612200204007
20	$\mathbb{Z}_{41}^2 \oplus \mathbb{Z}_{1051977795} \oplus \mathbb{Z}_{5259888975}$	9301454448259586320125

Table 2

$n$	$\text{Jac}(\widehat{Pr}(n))$	$ \text{Jac}(\widehat{Pr}(n)) $
3	$\mathbb{Z}_{24} \oplus \mathbb{Z}_{72}$	1728
4	$\mathbb{Z}_{15}^2 \oplus \mathbb{Z}_{105}$	23625
5	$\mathbb{Z}_{319} \oplus \mathbb{Z}_{957}$	305283
6	$\mathbb{Z}_8^2 \oplus \mathbb{Z}_{24} \oplus \mathbb{Z}_{2520}$	3870720
7	$\mathbb{Z}_{4031} \oplus \mathbb{Z}_{12093}$	48746883
8	$\mathbb{Z}_{105} \oplus \mathbb{Z}_{2415}^2$	612383625
9	$\mathbb{Z}_{50616} \oplus \mathbb{Z}_{151848}$	7685938368
10	$\mathbb{Z}_{30305} \oplus \mathbb{Z}_{3182025}$	96431267625
11	$\mathbb{Z}_{635009} \oplus \mathbb{Z}_{1905027}$	1209709290243
12	$\mathbb{Z}_{48} \oplus \mathbb{Z}_{720} \oplus \mathbb{Z}_{7920} \oplus \mathbb{Z}_{55440}$	15174770688000
13	$\mathbb{Z}_{7965569} \oplus \mathbb{Z}_{23896707}$	190350868481283
14	$\mathbb{Z}_{13}^2 \oplus \mathbb{Z}_{366821} \oplus \mathbb{Z}_{38516205}$	2387725428997545
15	$\mathbb{Z}_{99918456} \oplus \mathbb{Z}_{299755368}$	29951093548271808
16	$\mathbb{Z}_{105} \oplus \mathbb{Z}_{59817135}^2$	375699412158863625
17	$\mathbb{Z}_{1253353151} \oplus \mathbb{Z}_{3760059453}$	4712682363364886403
18	$\mathbb{Z}_8^2 \oplus \mathbb{Z}_{93791448} \oplus \mathbb{Z}_{9848102040}$	59114736024534650880
19	$\mathbb{Z}_{15721755199} \oplus \mathbb{Z}_{47165265597}$	741520759611850588803
20	$\mathbb{Z}_5 \oplus \mathbb{Z}_{75} \oplus \mathbb{Z}_{1882395075} \oplus \mathbb{Z}_{13176765525}$	9301454448258671015625

## 7. Examples

1°. THE WHEEL GRAPH  $W(n)$ . The graph  $W(n)$  is a cone over the cyclic graph  $C_n = C_n(1)$ . By Theorem 1, the number of spanning trees  $\tau(W(n))$  is equal

to the number of rooted spanning forests in  $C_n$  counting earlier in [14]. Hence,  $\tau(W(n)) = 2T_n(\frac{3}{2}) - 2$ . See also paper [2] for an alternating proof of this result.

By Theorem 3, Jacobian of the Wheel graph  $\text{Jac}(W(n))$  is isomorphic to the cokernel of linear operator  $\mathcal{A}^n - I_2$ , where  $\mathcal{A} = \{\{0, 1\}, \{-1, 3\}\}$  is a companion matrix of the Laurent polynomial  $3 - z - z^{-1}$ . Direct calculations leads to the well-known result [24]:  $\text{Jac}(W(n))$  is isomorphic to  $\mathbb{Z}_{F_n} \oplus \mathbb{Z}_{5F_n}$  if  $n$  is even, and  $\mathbb{Z}_{L_n} \oplus \mathbb{Z}_{L_n}$  if  $n$  is odd, where  $F_n$  and  $L_n$  are the Fibonacci and Lucas numbers respectively.

2°. THE CONE OVER THE MÖBIUS LADDER  $\widehat{M}(n)$ . Recall that the Möbius ladder  $M(n)$  is the circulant graph  $C_{2n}(1, n)$ . By Theorems 1 and 2 from [14], the number of spanning trees in the cone over the Möbius ladder  $\widehat{M}(n)$  can be found as

$$\tau(\widehat{M}(n)) = 4(T_n(3/2) - 1)(T_n(5/2) + 1).$$

The Jacobian of the cone over the Möbius ladder  $M(n)$  is isomorphic to the cokernel of the linear operator  $\mathcal{A}^n - 4I_4 + \mathcal{A} + \mathcal{A}^{-1}$ , where  $\mathcal{A}$  is a companion matrix of the Laurent polynomial  $(4 - z - z^{-1})^2 - 1$ .

Numerical calculation of  $\text{Jac}(\widehat{M}(n))$  and  $\tau(\widehat{M}(n)) = |\text{Jac}(\widehat{M}(n))|$  is given in Table 1.

3°. THE CONE OVER THE PRISM GRAPH  $\widehat{Pr}(n)$ . This graph is a cone over cobordism of two cyclic graphs  $C_n$ . By arguments similar to those from the proof of Theorem 2 in [14], the number of spanning trees of the cone over the prism graph  $\widehat{Pr}(n)$  is given by the formula

$$\tau(\widehat{Pr}(n)) = 4(T_n(3/2) - 1)(T_n(5/2) - 1).$$

By Theorem 5, the Jacobian of the cone over the prism graph  $Pr(n)$  is isomorphic to the cokernel of the linear operator  $\mathcal{A}^n - I_4$ , where  $\mathcal{A}$  is a companion matrix of the Laurent polynomial  $(4 - z - z^{-1})^2 - 1$ .

The numerical calculation of  $\text{Jac}(\widehat{Pr}(n))$  and  $\tau(\widehat{Pr}(n)) = |\text{Jac}(\widehat{Pr}(n))|$  is given in Table 2.

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