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ON SOME NEW ESTIMATES FOR INTEGRALS OF THE LUSIN'S SQUARE FUNCTION IN THE UNIT POLYDISK

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Abstract. The purpose of the note is to obtain new estimates for the quasinorm of Hardy's analytic classes of in the polydisk. We extend some classical onedimensional assertions to the case of several complex variables.

Our results more precisely provide direct new extention of some known one variable theorems concerning area integral to the case of simplest product domains namely the unit polydisk in $\mathbb{C}^n$. Let further D be a bounded or unbounded domain in $\mathbb{C}^n$. For example, tubular domain over symmetic cone or bounded pseudoconvex domain with smooth boundary. Our results can be probably extended to the case of products of such type complicated domains, namely even to $D \times \cdots \times D$. This can be probably done based on some approaches we suggested and used in this paper. On the other hand our results in simpler case namely in the unit polydisk may also have various interesting applications in complex function theory in the unit polydisk.

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In Euclidean space R^n the problem of finding equivalent quasinorm and exact estimates for the quasinorms of certain function spaces has been solved by many authors (see, for example, [1]). Recently, a large number of papers have appeared, where a similar problem was considered for analytic functions in the disk. Different equivalent quasinorms for analytic Besov type classes in disk $U = \{z : |z| < 1\}$ of complex plane $\mathbb{C}$ were derived in [2–5]. The papers [6, 7] gives several characteristics of the analytic classes in bidisk. Finally, at [6, 8] an equivalent quasinorm is indicated for space Lizorkin-Triebel in the polydisk. In this paper well-known estimates for the quasi-norm Hardy classes will be generalized to the case of the polydisk. It should be noted that the new functional spaces (see [9]) and their properties modified by us for the case of the polydisk play an essential role in the note.

1. New estimates of the quasinorm of Hardy classes in the polydisk

In this section we will extend some classical onedimensional assertions to Hardy spaces of several variables. We will present the following notation. Let U^n be the unit polydisk of C^n,

$$U^n = \{z \in C^n,\ |z_j| < 1,\ j = 1, \dots, n\}, \quad T^n = \{z \in C^n,\ |z_j| = 1,\ j = 1, \dots, n\},$$

$m_n(\xi)$ and $m_{2n}(\omega)$ be the normalized Lebesgue measures on T^n and U^n accordingly.

Let further

$$\widetilde{\Gamma}_\alpha(\xi) = \widetilde{\Gamma}_\alpha(\xi_1, \dots, \xi_n) = \Gamma_{\alpha_1}(\xi_1) \times \cdots \times \Gamma_{\alpha_n}(\xi_n),$$

where $\alpha_i > 0$, $\xi_i \in T$, $i = 1, \dots, n$,

$$\Gamma_{\alpha_i}(\xi_i) = \{z \in U : |1 - \bar{\xi}_i z| < \alpha_i(1 - |z|^2)\}, \quad i = 1, \dots, n;$$

and let also

$$I_{\overline{U}^n}(\xi_1, \dots, \xi_n, t_1, \dots, t_n) = \{z \in \overline{U}^n : |1 - z_1\bar{\xi}_1| < t_1, \dots, |1 - z_n\bar{\xi}_n| < t_n\};$$

$t_j > 0$, $j = 1, \dots, n$.

We omit α_j below sometimes dealing with $\Gamma_\alpha(\xi)$,

$$I_{U^n} = I_{\overline{U}^n} \cap U^n, \quad I_{T^n} = I_{\overline{U}^n} \cap T^n.$$

For a measurable f function in U^n we put

$$(A_q(f))(\xi) = \Bigg(\int\limits_{\widetilde{\Gamma}\alpha(\xi)} \frac{|f(z_1, \dots, z_n)|^q \, dm_{2n}(z)}{(1 - |z|)^2}\Bigg)^{1/q}, \quad q < \infty,$$

where $1 - |z| = \prod(1 - |z_j|)$;

$$(A_\infty(f))(\xi) = \sup_{z \in U^n} \{|f(z_1, \dots, z_n)|, \ z \in \widetilde{\Gamma}_\alpha(\xi)\}, \quad \xi \in T^n,$$

$$((C_q(f))(\xi))^q = \sup_{t_n} \frac{1}{|I_T(\xi_n, t_n)|} \int\limits_{I_U(\xi_n, t_n)} \frac{1}{(1 - |z_n|)} \cdots \times \sup_{t_1} \frac{1}{|I_T(\xi_1, t_1)|} \int\limits_{I_U(\xi_1, t_1)} \frac{|f(z_1, \dots, z_n)|^q}{(1 - |z_1|)} \, dm_{2n}(z),$$

$$\xi \in T^n, \quad \xi = (\xi_1, \dots, \xi_n), \quad t_j > 0, \quad j = 1, \dots, n.$$

Also let $H(U^n)$ and $H^p(U^n)$, $0 < p < \infty$, be the space of all holomorphic functions in U^n and the Hardy class in the polydisk accordingly,

$$H^p(U^n) = \Big\{f \in H(U^n) : \sup_r \int\limits_{T^n} |f(r\xi)|^p \, dm_n(\xi) < \infty\Big\}, \quad 0 < p \le \infty.$$

These are Banach spaces for all $p \ge 1$, and quazinormed spaces for other values of parameters.

Remark 1. The values given above in R^{n+1} were first introduced in [9] for the so-called new functional spaces in R^{n+1}.

A well-known statement of the theory of Hardy H^p spaces states that if a function f belongs to the Hardy class H^p, $0 < p < \infty$, then the quantity $S(f)$ is finite.

$$S(f) = \int\limits_T \Bigg(\int\limits_{\Gamma(\xi)} |D^k f(z)|^2 (1 - |z|)^{2k-2} \, dm_2(z)\Bigg)^{p/2} dm(\xi) < \infty.$$

D^k is a well-known differential operator of an analytic f function in the unit disk on the complex plane $\mathbb{C}$ (see, for example, [10, 11]). S operator is known as the Luzin area integral operator and the above statement about its boundedness in Hardy H^p classes, $0 < p < \infty$ in the disk U was established in [12]. In Theorem 1, relying on the multidimensional maximum theorem established in [12], two generalizations are given - direct polydisk analogues of this statement for $p < 2$.

In Theorem 2 for $p \geq 2$ this result will be generalized in two different ways at once. Let D^α be the fractional derivative of the f function, $\alpha \geq 0$, $D^\alpha : H(U^n) \to H(U^n)$,

$$(D^\alpha f)(z_1, \ldots, z_n) = \sum_{|k| \geq 0} (k+1)^\alpha \alpha_{k_1 \ldots k_n} z_1^{k_1} \ldots z_n^{k_n}, \quad (z_1, \ldots, z_n) \in U^n,$$

$$(k+1) = \prod_{j=1}^{n} (k_j + 1),$$

$$f(z_1, \ldots, z_n) = \sum_{k_1 \ldots k_n \geq 0}^{+\infty} a_{k_1 \ldots k_n} z_1^{k_1} \ldots z_n^{k_n},$$

(see [5, 11, 13]).

We define fractional derivative also for negative indexes, namely we put

$$D^{-\alpha}(D^\alpha)(f) = f, \quad \alpha \geq 0.$$

Let further $C_1, C_2, C(n)$ be various positive constants.

Everywhere below the notation $A < B$ denotes that there is the constant $C > 0$ such that $A \leq CB$, the notation $A = B$ denotes that there are the constants $C_1 > 0$ and $C_2 > 0$ such that $C_2 B \leq A \leq C_1 B$.

Theorem 1. (A) *Let $f \in H(U^n)$. Assume that*

$$F(z_1, \ldots, z_n) = \sum_{k_1 \ldots k_n \geq 0}^{+\infty} (k_2 + 1) \ldots (k_n + 1) a_{k_1 \ldots k_n} z_1^{k_1} \ldots z_n^{k_n} \tag{1}$$

is in class $H^p(U^n)$, $0 < p < 2$,

$$S_n(f) = \left(\int_{T^n} \left(\int_{\Gamma(\xi_1)} \cdots \int_{\Gamma(\xi_n)} |Df(z_1, \ldots, z_n)|^2 \, dm_{2n}(z) \right)^{p/2} dm_n(\xi) \right)^{1/p}.$$

Then next estimates are correct

$$C_1 \|D^{-\varepsilon} f\|_{H^p(U^n)} \leq S_n(f) \leq C_2 \|f\|_{H^p(U^n)}, \quad 0 < p < 2, \tag{2}$$

where ε is arbitrary positive number and $D(f) = D^1(f)$.

(B) *Let $0 < p < 2$, $f \in H(U^n)$. Then next estimates are correct*

$$C_1 \|D^{-\varepsilon} f\|_{H^p(U^n)} \leq \widetilde{S}_n(f) \leq C_2 \|f\|_{H^p(U^n)},$$

where ε is arbitrary positive number and

$$\widetilde{S}_n(f) = \Bigg(\int\limits_T \Bigg(\int\limits_{\Gamma(\xi_n)} \dots \Bigg(\int\limits_T \Bigg(\int\limits_{\Gamma(\xi_1)} |Df|^2 \, dm_2(z_1)\Bigg)^{\frac{p}{2}}$$

$$\times \, dm(\xi_1)\Bigg)^{\frac{2}{p}} \dots dm_2(z_n)\Bigg)^{\frac{p}{2}} dm(\xi_n)\Bigg)^{\frac{1}{p}}. \quad (3)$$

Remark 2. For $n = 1$ the condition (1) disappears and the statements of the theorem coincide and are well known (see [5, 12]).

Remark 3. It will be seen from the proof that by slightly modifying the reasoning in a similar way a somewhat more general result for Hardy–Sobolev classes can be proved. They are defined as follows:

$$H^p_\alpha = \{f \in H(U^n) : \|D^\alpha f\|_{H^p} < \infty\},$$

$\alpha \geq 0$, $P \in (0, \infty)$.

Proof of Theorem 1. The following lemma will be needed.

Lemma 1. *Let $f \in H(U^n)$, then next estimates are correct*

$$\int\limits_{T^n} (\sup_{z \in \widetilde{\Gamma}(\xi)} |D^\alpha f(z)|(1-|z|)^\alpha)^p \, dm_n(\xi) < C\|f\|^p_{H^p}, \quad 0 < p < +\infty, \ \alpha \geq 0; \quad (4)$$

$$\underbrace{\int\limits_T \cdots \int\limits_T}_{n-1} \int\limits_U |Df(z_1, \xi_2, \dots, \xi_n)|^2$$

$$\times \, |z_1 F(z_1, \xi_2, \dots, \xi_n)|^{p-2} (1-|z_1|) \, dm_2(z_1) dm_1(\xi_2) \dots dm_1(\xi_n) \leq \|F\|^p_{H^p(U^n)}, \quad (5)$$

$0 < p < \infty$;

$$\int\limits_{U^n} |z_1 F(z_1, \dots, z_n)|^{p-2} |Df(z_1, \dots, z_n)|^2 (1-|z_1|) \dots (1-|z_n|) \, dm_{2n}(z_1, \dots, z_n)$$

$$\leq \int\limits_U \cdots \int\limits_U \int\limits_T |F(\xi_1, z_2, \dots, z_n)|^p (1-|z_2|) \dots (1-|z_n|) \, dm(\xi) dm_2(z_2), \dots, dm_2(z_n).$$

Here and hereafter F is defined by (1) $0 < p < \infty$. The first inequality is proved in [14] and relies on the multivariable maximal theorem proved in [12, 15]. The second and third inequalities are not difficult to establish directly relying on one consequence of Green's formula (see [12, 15]):

$$\int\limits_{-\pi}^{\pi} W(e^{i\theta}) \, d\theta = \int\limits_U \left(\log \frac{1}{|z|}\right) \Delta W(z) \, dxdy,$$

where Δ is the Laplace operator, $W \in C^2(U \cup T)$, $W(0) = 0$.

Selecting the function $(|zF(\rho z, z_2, \dots, z_n)|^2 + \varepsilon)^{\frac{p}{2}}$, $0 < p < 1$, $\varepsilon > 0$ as W and applying this formula we get

$$\int_{-\pi}^{\pi} (|F(\rho e^{i\theta_1}, z_2, \dots, z_n)|^2 + \varepsilon)^{\frac{p}{2}}\, d\theta_1$$
$$= \int_U \log \frac{1}{|z|} \Delta(|zF(\rho z, z_2, \dots, z_n)|^2 + \varepsilon)^{\frac{p}{2}}\, dm_2(z) + 2\pi\varepsilon^{\frac{p}{2}};$$

passing to the limit at $\varepsilon \to 0$ and given that $\Delta F = 4\frac{\partial^2 F}{\partial z \partial \bar{z}}$ we get

$$\int_{-\pi}^{\pi} |F(\rho e^{i\theta_1}, z_2, \dots, z_n)|^p\, d\theta_1 = \frac{p^2\rho^2}{4} \int_U |zF(\rho z, z_2, \dots, z_n)|^{p-2}$$
$$\times |Df(\rho z, z_2, \dots, z_n)|^2 \log \frac{1}{|z|}\, dm_2(z);$$

Now the inequalities (2) and (3) are easily obtained by relying on this proportion. We can integrate both parts either by T^{n-1} with measure $dm(\xi_2) \dots dm(\xi_n)$ or by U^{n-1} with measure $(1 - |z_2|) \dots (1 - |z_n|)\, dm_2(z_2) \dots dm_2(z_n)$ appropriately. From here we have for $0 < p < 2$

$$\int_{U^n} |F(z_1, \dots, z_n)|^{p-2} |Df(z_1, \dots, z_n)|^2 (1 - |z_1|) \dots (1 - |z_n|)\, dm_2(z)$$
$$\le \sup_{|z_1|, \dots, |z_n|} \Bigg(\int_{T^n} |F(z_1, \dots, z_n)|^p\, dm_n(\xi) (1 - |z_2|)^p \dots$$
$$\times (1 - |z_n|)^p \left(\int_0^1 (1 - r)^{1-p} dr \right)^{n-1} \Bigg) < C\|f\|^p_{H^p}, \quad 0 < p < 2. \quad (6)$$

Let us first establish that condition (2) is necessary for f function to belong to the Hardy H^p class. Let $0 < p < 2$, then we have

$$S^p_n(f) < C_1 \int_{T^n} (\sup_{z \in \Gamma(\xi)} |F(z_1, \dots, z_n)|^{\frac{(2-p)p}{2}})$$
$$\times \left(\int_{\Gamma(\xi_1)} \cdots \int_{\Gamma(\xi_n)} \frac{|Df|^2\, dm_{2n}(z)}{|F(z)|^{2-p}} \right)^{p/2} dm_n(\xi).$$

Recall that F is defined in equality (1). Next we apply the Holder's inequality with the exponent $p' = \frac{2}{2-p}, q' = \frac{2}{p}$ and (7') and we obtain

$$S^p_n(f) < C_2 \int_{T^n} (\sup_{z \in \Gamma(\xi)} |F(z)|^p\, dm_n(\xi))^{\frac{2-p}{2}}$$
$$\times \left(\int_{U^n} |F(z)|^{2-p} |Df(z)|^2 (1 - |z|)\, dm_{2n}(z) \right)^{p/2}.$$

Next we use the condition of the Theorem and estimates (4) and (6) respectively, we get

$$S_n^p(f) < C\|F\|_{H^p(U^n)}^{\frac{(2-p)p}{2}}\|f\|_{H^p(U^n)}^{\frac{p^2}{2}}.$$

From here it is easy to get $S_n^p(f) < C\|f\|_{H^p}^p$, $0 < p < 2$.

Let us now prove the left inequality.

Let $p \le 1$. Then from known estimate (see [11]) we have

$$\left(\int\limits_{U^n} |f(z)|(1-|z|)^\alpha\, dm_{2n}(z)\right)^p < C_3 \int\limits_{U^n} |f(z)|^p(1-|z|)^{\alpha p+2p-2}\, dm_{2n}(z), \qquad (6')$$

$p \le 1$, $(\alpha+2)p > 1$, $f \in H(U^n)$.

Hence we deduce

$$\int\limits_{T^n} |f(r_1\xi_1,\dots,r_n\xi_n)|^p\, dm_n(\xi) <$$

$$< C_4 \int\limits_{T^n}\left(\int\limits_{U^n} |Df(w)|^p \left|D^\alpha\left(\frac{1}{1-\bar{w}z}\right)\right|^p (1-|w|)^{\alpha p+2p-2})\, dm_{2n}(w)\right) dm_n(\xi)$$

$$< C_5 \int\limits_{U^n} |Df(w)|^p(1-|w|)^{p-1}\, dm_{2n}(w), \qquad (6'')$$

$r_j \in (0,1)$, $j = 1,\dots,n$.

For $1 < p \le 2$ the estimate $(6'')$ can be obtained similarly using instead $(6')$ estimate

$$\left(\int\limits_{U^n} \frac{|f(z)|(1-|z|)^\alpha}{|1-\bar{w}z|^{\beta+2}}\, dm_{2n}(z)\right)^p < \int\limits_{U^n} \frac{|f(z)|^p(1-|z|)^{\alpha p}(1-|w|)^{-\varepsilon p}}{|1-\bar{w}z|^{(\beta-\varepsilon)p+2}}\, dm_{2n}(z),$$

where $\alpha > -1$, $\varepsilon > 0$, $\beta > 0$, which is easily obtained from the Holder's inequality. Next, we will need the following estimates, they are well-known at $n = 1$ (see [9]),

$$\left|\int\limits_{U^n} \frac{f(z)g(\bar{z})}{(1-|z|)}\, dm_{2n}(z)\right| < C_6 \int\limits_{T^n} (A_{q'}(f)(\xi))(C_q(g)(\xi))\, dm_n(\xi) \qquad (7)$$

$$\int\limits_{U^n} |\Phi(z)|(1-|z|)^\alpha\, dm_{2n}(z) < C_7 \int\limits_{T^n}\int\limits_{\Gamma(\xi)} |\Phi(z)|(1-|z|)^{\alpha-1}\, dm_{2n}(z) dm_n(\xi). \qquad (7')$$

Estimates (7) and (7') are not difficult to obtain from one-dimensional version by sequential application over each variable.

Based on (7') and (6") we deduce

$$||D^{-\varepsilon}f||_{H^p}^p \le C \int\limits_{U^n} |Df(w)|^p(1-|w|)^{p-1}(1-|w|)^\varepsilon\, dm_{2n}(w)$$

$$\le C \int\limits_{T^n}\left(\int\limits_{\Gamma(\xi_1)} \cdots \int\limits_{\Gamma(\xi_n)} \frac{(|Df(w)|^p(1-|w|)^p)^{(2/p)}}{(1-|w|)^2}\, dm_{2n}(w)\right)^{p/2}$$

$$\times \Bigg(\int\limits_{\Gamma(\xi_1)} \cdots \int\limits_{\Gamma(\xi_n)} (1-|w|)^{-2+\varepsilon(2/p)'}\, dm_{2n}(w) \Bigg)^{1/(2/p)'} dm_n(\xi) \le (S_n(f))^p.$$

The first part of the theorem is established.

Remark 4. Note that above we (right inequality) have modified the reasoning applicable to $n=1$, previously by various authors (see, for example, [10, 12, 13, 15]).

An essential role in our reasoning is played by the maximum theorem in U^n, established in [12, 14], the integrals of the squares by Luzin and values $A_q(f)$ and $C_q(f)$ in U^n chosen in a suitable way.

We establish the second statement of the theorem. Applying the estimate (7′) for each variable separately with the exponent $2/p > 1$ and repeating the above reasoning for each variable, we obtain

$$\int\limits_{T^n} |D^{-\varepsilon} f(r\xi)|^p\, dm_n(\xi) \le \int\limits_{U^n} \frac{|Df(w)|^p(1-|w|)^{p+\varepsilon}}{(1-|w|)}\, dm_{2n}(w)$$

$$< C_8 \int\limits_{U^{n-1}} \int\limits_{T} \Bigg(\int\limits_{\Gamma(\xi_1)} |Df(w)|^2\, dm_2(w_1) \Bigg)^{p/2} dm(\xi_1)$$

$$\times \frac{((1-|w_2|)\ldots(1-|w_n|))^{p+\varepsilon}}{(1-|w_2|)\ldots(1-|w_n|)}\, dm_{2n-2}(w) < (\widetilde{S}_n(f))^p.$$

Let us now prove the right inequality. For this we modify the arguments given in proving of the corresponding inequality (A).

Let

$$F = \frac{\partial^{n-1} w_2 \ldots w_n f}{\partial w_2 \ldots \partial w_n}.$$

Reasoning similarly we have

$$\widetilde{S}_1 = \int\limits_{T} \Bigg(\int\limits_{\Gamma(\xi_1)} \left| \frac{\partial^n z_1 \ldots z_n f(z)}{\partial z_1 \ldots \partial z_n} \right|^2 dm_2(z_1) \Bigg)^{p/2} dm(\xi_1)$$

$$< C_9 \Bigg(\int\limits_{T} \sup_{z_1 \in \Gamma(\xi_1)} |F(z)|^p\, dm(\xi_1) \Bigg)^{\frac{2-p}{2}}$$

$$\cdot \Bigg(\int\limits_{U} |F(z)|^{p-2} \left| \frac{\partial^n (z_1 \ldots z_n f)}{\partial z_1 \ldots \partial z_n} \right|^2 (1-|z_1|)\, dm_2(z_1) \Bigg)^{p/2}$$

$$< C_{10} \left\| \frac{\partial^{n-1}(z_2 \ldots z_n f(z))}{\partial z_2 \ldots \partial z_n} \right\|_{H^p_{z_1}(U)}^{p(1-p/2)} \left\| \frac{\partial^{n-1}(z_2 \ldots z_n f(z))}{\partial z_2 \ldots \partial z_n} \right\|_{H^p_{z_1}(U)}^{p\cdot(p/2)}$$

$$= \left\| \frac{\partial^{n-1}(z_2 \ldots z_n f(z))}{\partial z_2 \ldots \partial z_n} \right\|_{H^p_{z_1}(U)}^{p}.$$

We integrate the last inequality by $\Gamma(\xi_2)$ and T by z_2. We apply Minkowski's inequality, Fubini's theorem and repeat the reasoning for $\widetilde{S}_1$ on the variable z_2.

Repeating this procedure $n-2$ times will come to the desired result. Theorem 1 is fully proved.

For the f function, $f \in H(U^n)$ denote

$$G(f,\alpha,p,\gamma) = \int\limits_{T^n} \Bigg(\int\limits_{\Gamma(\xi_1)} \cdots \int\limits_{\Gamma(\xi_n)} |D^k f(z)|^{\alpha+2} \times (1-|z|)^{\alpha/p+(\alpha+2)k-2-\gamma}\, dm_{2n}(z) \Bigg)^{p/2} dm_n(\xi).$$

The following theorem generalizes the known one-dimensional estimates from below for the Hardy class norm $H^p(U)$ for $p \geq 2$ (in two directions at once). This theorem also establishes the ε-accuracy of this estimate.

Theorem 2. *Let $p \geq 2$, $f \in H^p(U^n)$, $0 \leq \alpha < p$. Then*

$$G(f,\alpha,p,o) \leq C_2 \|f\|^p_{H^p}. \tag{8}$$

This estimate is accurate in the following sense. For all $\varepsilon, \varepsilon > 1-2/p$ there will always be α number, $\alpha \geq 0$, such that the inequality is true

$$\|f\|^p_{H^p} \leq C_1(G(f,\alpha,p,\varepsilon)),$$

C_1, C_2 there are some positive constants.

The proof of Theorem 2. The theorem we prove in bidisk. The General case is exhausted similarly. The proof of the estimate (8) relies on the ratio (7) and (7′). Indeed, using duality arguments and the formula

$$\int\limits_T \Bigg(\int\limits_{\Gamma(\xi)} g(w)\, dm_2(w) \Bigg) \psi(\xi)\, dm(\xi) = \int\limits_U g(z) \int\limits_T \lambda_{\Gamma(\xi)}(z_1)\psi(\xi)\, dm(\xi) dm_2(z)$$

(λ is a characteristic function of $\Gamma(\xi)$) for each variable separately we deduce

$$G(f,\alpha,p,o) = M_2 \leq \int\limits_U \int\limits_U |D^k f(z)|^{\alpha+2}(1-|z|)^t \times \int\limits_T \lambda_{\Gamma(\xi_2)}(z_2) \Bigg(\int\limits_T \lambda_{\Gamma(\xi_1)}(z_1)\psi(\xi_1,\xi_2)\, dm(\xi_1) \Bigg) dm(\xi_2) dm_2(z_1) dm_2(z_2),$$

where $\psi(\xi_1,\xi_2) \in L^q(T^2)$, $q = \big(\frac{p}{2}\big)'$, $t = \alpha\big(\frac{1}{p}+k\big) + 2k - 2$.

Next given the estimate (7) with the exponent $q = \infty, q' = 1$ we deduce the inequality:

$$M_2 \leq C_1 \int\limits_T \int\limits_T \sup_{z_1 \in \Gamma(\tilde{\xi}_1)} \sup_{z_2 \in \Gamma(\tilde{\xi}_2)} |D^k f(z)|^2 (1-|z|)^{2k} \times \sup_{z_1 \in \Gamma(\tilde{\xi}_1)} \sup_{z_2 \in \Gamma(\tilde{\xi}_2)} \frac{1}{(1-|z_1|)(1-|z_2|)} \int\limits_T \lambda_{\Gamma(\xi_2)}(z_2) \Bigg(\int\limits_T \lambda_{\Gamma(\xi_1)}(z)\psi(\xi_1,\xi_2)\, dm(\xi_1) \Bigg)$$

$$\times\,(|D^k f(z)|^\alpha (1-|z|)^{\alpha(1/p+k)})\,dm(\tilde{\xi_1}),dm(\tilde{\xi_2})$$
$$\leq \|C_1((1-|z|)^{\alpha(1/p+k)}|D^k f(z)|^\alpha)\|_{L^\infty(T^2)}$$
$$\times\int\limits_T\int\limits_T\{\sup_{z_1\in\Gamma(\tilde{\xi_1})}\sup_{z_2\in\Gamma(\tilde{\xi_2})}(|D^k f(z)|^2(1-|z|)^{2k})\}(M(\psi)(\tilde{\xi_1},\tilde{\xi_2}))\,dm(\tilde{\xi_1})dm(\tilde{\xi_2}),\quad (8')$$

where $M(f)$ is a maximal function of Hardy-Littlewood;

$$M(f)(\tilde{\xi_1},\tilde{\xi_2})=\sup_{t_1>0}\frac{1}{|I_{\tilde{\xi}_1,t}|}\int\limits_{I_{\xi_1,t}}\sup_{t_2>0}\left[\frac{1}{|I_{\tilde{\xi}_2,t}|}\int\limits_{I_{\xi_2,t}}|\psi(\varphi_1,\varphi_2)|\,dm(\varphi_1)\right]dm(\varphi_2.)\quad (9)$$

To estimate the first multiplier, it is sufficient to apply the Holder inequality with the exponent $p/2$ and two maximal theorems (in the polydisk), one of which was mentioned above (see the estimate (4)) and the latter theorem was established in [14]. The second estimate on the action of the Hardy–Littlewood operator is derived by applying a one-dimensional result (see [12]) by each variable.

Now let's estimate the second multiplier. We have

$$A=(|D^k f(z)|^\alpha(1-|z|)^{k\alpha+\alpha/p})(\xi_1,\xi_2)\leq\sup_{t_2}\frac{1}{|I_T(\xi_2,t_2)|}\int\limits_{I_U(\xi_2,t_2)}(1-|z_2|)^{k\alpha+\alpha/p-1}$$
$$\times\sup_{t_1}\frac{1}{|I_T(\xi_1,t_1)|}\int\limits_{I_U(\xi_1,t_1)}|D^k f(z)|^\alpha(1-|z_1|)^{k\alpha+\alpha/p-1}\,dm_2(z_1)dm_2(z_2).\quad (10)$$

We will evaluate "the inner sup" (use the Holder inequality with the exponent $(p/\alpha)'$) for $\int\limits_{|1-\bar{\xi}_1\eta_1|<t_1}|D^k f(z)|^\alpha\,dm(\xi_1)$:

$$\sup_{t_1}(t_1^{-1})\int\limits_{1-t_1}^{1}\int\limits_{|1-\bar{\xi}_1\eta_1|<t_1}|D^k f(z)|^\alpha(1-|z_1|)^{k\alpha+\alpha/p-1}\,dm_2(z_1)$$
$$\sup_{t_1}(t_1^{-1})\int\limits_{1-t_1}^{1}M_p^p(D^k f,|z_1|)^{\alpha/p}t_1^{\frac{1}{(p/\alpha)'}}(1-|z_1|)^{k\alpha+\alpha/p-1}\,dm|z_1|$$
$$<C\left(\int\limits_T|D^k_{(z_2)}f(\xi_1,|z_2|\xi_2)|^p\,dm(\xi_1)\right)^{\alpha/p}.$$

The second multiplier is evaluated similarly. So,

$$A\leq\widetilde{C}_1\|f\|^p_{H^p}.\quad (11)$$

Remark 5. The above inequality (8) is well known for $n=1$ (see [12]).

To establish the second inequality, we use the known one-dimensional embedding (see [5, 9, 10, 12, 13, 16]):

$$A^r_s(U)\subset H^p(U),\quad r<p,\quad s-\frac{1}{r}=-\frac{1}{p},\quad p>2,\quad r<\alpha+2,\quad (12)$$

where A_s^r is analytic Besov's class in U,

$$A_s^r(U) = \left\{ f \in H(U) : \int\limits_U |D^k f(z)|^r (1-|z|)^{(k-s)r-1}\, dm_2(z) < \infty \right\}.$$

Using embedding (12) (by each variable) and estimate (7') with an exponent $\frac{\alpha+2}{r}$, we deduce

$$\|f\|_{H^p}^p \le \int\limits_{T^n} \left(\int\limits_{\Gamma(\xi)} |D^k f(z)|^{\alpha+2} (1-|z|)^{\alpha/p+(\alpha+2)k-2-\varepsilon}\, dm_2(z) \right)^{\frac{p}{2}} dm_n(\xi)$$
$$\times \|C_{(\frac{\alpha+2}{r})}(1-|z|)^{\gamma}\|_{L^\infty(T^n)}, \quad (13)$$

where

$$\left(\frac{\alpha+2}{r}\right)' \gamma = \left(\frac{\alpha+2}{\alpha+2-r}\right) \left\{ (k-s)r - \frac{r}{\alpha+2}\left(-\varepsilon + \frac{\alpha}{p} + (\alpha+2)k\right) \right\}. \quad (14)$$

It remains to be noted that from the conditions of the Theorem and the obtained relations it is possible to choose α such that $\gamma > 0$, and the latter is sufficient for the finiteness of the second multiplier in (13). Theorem is proved.

REFERENCES

1. *Triebel Kh.* Theoriya funktsionalnykh prostranstv (The theory of functional spaces), Moscow, (1986).
2. *Dyakonov K. M.* Besov spaces and outer functions. Michigan Math. J. **45** (1998), No. 1, 143–157.
3. *Dyakonov K. M.* Equivalent norms on Lipschitz-type spaces of holomorphic functions. Acta Math., 178 (1997), 143–167.
4. *Yoneda R.* Characterizations of Bloch space and Besov spaces by oscillations. Hokkaido Math. J. 29 (2000), no. 2, 409–451.
5. *Zhu K.* Operator theory in function spaces. New York: Springer, (1990).
6. *Shamoyan R. F.* O prostranstvakh golomorfnykh v polikruge funktsiy tipa Lizorkina–Tribelya (On spaces of holomorphic functions in polydisk Lizorkin–Tribel type), Izv. NAN Arm., 3 (2002), 57–78.
7. *Shamoyan R. F.* On the quasinorm of holomorphic functions from classes Lizorkin-Tribel in subostov, Symposium "Fourier Series and their applications" (in Russian), (2002), 54–55.
8. *Guliyev V. S., Lizorkin P. I.* Classes of holomorphic and harmonic functions in a polycircle in connection with their boundary values, Research on the theory of differentiable functions of many variables and its applications. Proc. Steklov Inst. Math., 204 (1994), 117–135.
9. *Coifman R., Meyer Y., Stein E.* Some new functional spaces and their application to harmonic analysis. Journal of Functional Analysis, 62:2 (1985), 304–335.
10. *Cohn W., Verbitsky I.* Factorization of Tent spaces and Hankel operators. Journal of Functional Analysis, 175 (2000), 308–329.
11. *Djrbashian M. M., Shamoian F. A.* Topics in the theory of A^p_α spaces, Leipzig, Teubner-Verl., (1988).
12. *Zygmund A.* Trigonometric series Vols. I, II (2nd ed.), Cambridge University Press,(1959).
13. *Cohn W.* Bergman projections and operators on Hardy spaces. Funct. Anal. J., 144 (1997), 1–19.
14. *Gvaradze M.* Mnozhiteli odnogo klassa analiticheskikh funktsiy, opredelennykh na polidiske, Tr. Tbil. Mat. In-ta (In Russian), 66 (1980), 15–21.
15. *Koosis J.B.* Bounded analytic functions. Academic Press, Orlando, Fla, (1981).

16. *Cohn W.* A factorization theorem for the derivative of a function in H^p. Proc. Amer. Math. Soc. 127:2 (1999), 507–517.

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