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ON ONE APPLICATION OF THE ZYGMUND—MARCINKIEWICZ THEOREM

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Abstract. In this paper we aim to generalize results obtained in the framework of fractional calculus due to reformulating them in terms of operator theory. In its own turn, the achieved generalization allows us to spread the obtained technique on practical problems connected with various physical and chemical processes. More precisely, a class of existence and uniqueness theorems is covered, the most remarkable representative of which is the existence and uniqueness theorem for the Abel equation in a weighted Lebesgue space. The method of proof corresponding to the uniqueness part is worth noticing separately: it reveals properties of the operator as well as properties of the space into which it acts and emphasizes their relationship.

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1. Introduction

To write this paper, we were firstly motivated by the boundary value problems connected with various physical and chemical processes: filtration of liquid and gas in highly porous fractal medium; heat exchange processes in medium with fractal structure and memory; casual walks of a point particle that starts moving from the origin by self-similar fractal set; oscillator motion under the action of elastic forces which is characteristic for viscoelastic media, etc. It is worth noticing that the attention of many engineers are being attracted to the following processes: dielectric polarizations [1], electrochemical processes [2–4], colored noises [5], chaos [6]. The special interest is devoted to the viscoelastic materials [7–10]. As it is well-known, to describe these processes we have to involve the theory of differential equations of fractional order and such notions as the Riemann–Liouville, Marchaud, Weyl, Kipriyanov fractional derivatives, which have applications to the physical and chemical processes listed above. For instance, the model of the so-called intensification of scattering processes is based on the fractional telegraph equation which was studied in the papers [11, 12]. In its own turn, the foundation of models describing the processes listed above can be obtained by virtue of fractional calculus methods, the central point of which is a concept of the Riemann–Liouville operator acting in a weighted Lebesgue space. In particular, we would like to point out the reader attention to the boundary value problems for the second order differential operator with the Riemann–Liouville fractional derivative in final terms. Many papers were devoted to this question, for instance [13–16]. It is quite reasonable that the variety of fractional derivative senses mentioned above creates a motivation to consider abstract methods in order to solve concrete problems without inventing an additional technique in each case. In its own turn, the operator theory methods play an important role in applications and need not any of special advertising. Having forced by these reasons, we deal with mapping theorems for operators acting on Banach spaces in order to obtain afterwards the desired results applicable to integral operators. We also note that our interest was inspired by lots of previously known results related to mapping theorems for fractional integral operators obtained by mathematicians such as B. S. Rubin [17–19], B. G. Vakulov [20], S. G. Samko [21, 22], N. K. Karapetyants [23, 24]. In the contrary to the central aim of these works, most of them are devoted to mapping properties of an operator that acts from one Lebesgue space into another one, the main idea of this paper is to find sufficient conditions for solvability of operator equations in abstract Banach space. We have chosen integral operators as a concrete material (see [25, 26]) to create an abstract model. Thus the prospective results must be harmoniously connected with applications and look like novel on the account of the novelty of the last ones.

2. Preliminaries

We consider a pair of spaces with a finite measure $(\Omega, \mathscr{F}, \mu_i)$, $i = 0, 1$. The corresponding Banach spaces of real functions defined on a set Ω are denoted by $L_p(\Omega, \mu_i)$, $1 < p < \infty$. As usual, we assume that $L_p(\Omega, \mu_i)$ are reflexive and the Riesz representation theorem is true. The dual normed spaces are denoted by $L_{p'}(\Omega, \mu_i)$ respectively, where $1/p + 1/p' = 1$. Denote by C a positive real constant. Suppose that there exists a set $\{e_n\}_0^\infty \subset L_p(\Omega, \mu_0) \cap L_p(\Omega, \mu_1)$ that has a basis property in the spaces $L_p(\Omega, \mu_1), L_{p'}(\Omega, \mu_1)$, we also assume that

$$\int\limits_{\Omega} e_m e_n \, d\mu_1 = \delta_{mn}, \quad m, n = 0, 1, \ldots,$$

where δ_{mn} is the Kronecker delta. Under this assumption, it is quite reasonable to involve the ordinary notations

$$S_k f := \sum_{n=0}^{k} f_n e_n, \quad f \in L_p(\Omega, \mu_1),\ f_n := \int\limits_{\Omega} f e_n d\mu_1,\ k = 0, 1, \ldots.$$

We also assume that there exist constants $\nu \in (2, \infty], \zeta \in [0, \infty)$ such that

$$\left(\int\limits_{\Omega} |e_n|^\nu \, d\mu_1 \right)^{1/\nu} \leq C n^\zeta,$$

where the expression above is understood as $\|e_n\|_{L_\infty(\Omega,\mu_1)}$, if $\nu = \infty$. We need the following essential assumption. The measure μ_1 and the domain Ω are such that Hypothesis 1 (see Appendix) is correct. Note that in the case where μ_1 is a Lebesgue measure and $\Omega = (a,b) \subset \mathbb{R}$ we have the fact that Hypothesis 1 is correct and well-known as the Zygmund–Marcinkiewicz theorem (see Theorem 3 in [27]), another example of a measure of this type was considered in [26]. Let A, B be a pair of linear operators bounded in $L_p(\Omega,\mu_0), L_p(\Omega,\mu_1)$ respectively; moreover, both operators have the same restriction on $L_p(\Omega,\mu_0)\cap L_p(\Omega,\mu_1)$. Denote by $\mathfrak{D}(B)$, $\mathfrak{R}(B)$ the domain of definition and the range of the operator B respectively. Assume that the operator A^{-1} is defined on $\{e_n\}_0^\infty$ and denote the following functionals by

$$(e_m, Ae_n)_{\mu_1} =: A_{mn}, \quad (e_m, A^{-1}e_n)_{\mu_1} =: A'_{mn}. \tag{1}$$

Here and further we use the following notations

$$(f,g)_{\mu_i} := \int\limits_{\Omega} fg\, d\mu_i, \quad (f,g)_{\mu_0,n} := \int\limits_{\Omega_n} fg\, d\mu_0.$$

3. Main results

The following theorems are formulated in terms of Hypothesis 1 (see Appendix) and provide a description of mapping properties of a quite wide operator class including fractional integral operators.

Theorem 1. *Suppose $\psi \in L_p(\Omega,\mu_1)$, $2 \le p < \infty$,*

$$\sum_{n=0}^{\infty} \psi_n A_{mn} = O(m^{-\lambda}), \quad \lambda \ge 0;$$

then

$$A\psi = f \in L_q(\Omega,\mu_1), \tag{2}$$

where $q = p$, if $0 \le \lambda \le 1/2$; q is an arbitrary large number satisfying

$$q < \min\left\{\nu, \frac{\nu(2\zeta+1)}{\nu(\zeta+1-\lambda)+2\lambda-1}\right\}, \tag{3}$$

if $1/2 < \lambda < [\nu(\zeta+1)-1]/(\nu-2)$; and q is an arbitrary large number satisfying $q < \nu$, if $\lambda \ge [\nu(\zeta+1)-1]/(\nu-2)$, where the parameters ν, ζ are defined in section 2. Moreover, if $\lambda > 1/2$, then f is represented by a convergent in $L_q(\Omega,\mu_1)$ series

$$f = \sum_{m=0}^{\infty} e_m \sum_{n=0}^{\infty} \psi_n A_{mn}. \tag{4}$$

This theorem can be formulated in a matrix form

$$A \times \psi = f \quad \sim \quad \begin{pmatrix} A_{00} & A_{01} & \dots \\ A_{10} & A_{11} & \dots \\ \cdot & & \\ \cdot & & \\ \cdot & & \dots \end{pmatrix} \times \begin{pmatrix} \psi_0 \\ \psi_1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}.$$

PROOF. Note, that in accordance with the basis property of $\{e_n\}_0^\infty$, we have

$$\sum_{n=0}^{l} \psi_n e_n \stackrel{L_p(\Omega,\mu_1)}{\longrightarrow} \psi \in L_p(\Omega,\mu_1), \quad l \to \infty.$$

Since the operator A is bounded, then

$$\sum_{n=0}^{l} \psi_n A e_n \stackrel{L_p(\Omega,\mu_1)}{\longrightarrow} A\left(\sum_{n=0}^{\infty} \psi_n e_n\right) = A\psi, \quad l \to \infty.$$

Hence by virtue of the inner product continuity property, we get

$$\sum_{n=0}^{l} \psi_n (Ae_n, e_m)_{\mu_1} \longrightarrow (A\psi, e_m)_{\mu_1}, \quad l \to \infty.$$

Using denotations (1), we have

$$f_m = (A\psi, e_m)_{\mu_1} = \sum_{n=0}^{\infty} \psi_n A_{mn}.$$

Since $0 \le \lambda \le 1/2$, then it is not hard to prove that

$$\frac{\nu(2\zeta+1)}{\nu(\zeta+1-\lambda)+2\lambda-1} \le 2.$$

This implies that we cannot use condition (3) to obtain the additional information on the function f. However, note that in any case by virtue of the conditions imposed on the operator A, we have $p = q$. Using condition (3), by simple calculation in the case $1/2 < \lambda < [\nu(\zeta+1)-1]/(\nu-2)$, we obtain

$$\zeta\frac{\nu(q-2)}{\nu-2} + \frac{\nu-1}{\nu-2}(q-2) - \lambda q < -1. \tag{5}$$

This implies that corresponding series (16), where we put $M_n := n^\zeta$, is convergent. By virtue of this fact, having applied Hypothesis 1, we obtain (2). Now assume that $[\nu(\zeta+1)-1]/(\nu-2) \le \lambda$, then consider relation (5) and let us prove that it is fulfilled for $2 \le q < \infty$. By easy calculations, we can rewrite (5) in the following form

$$q\left\{\frac{\zeta\nu}{\nu-2} + \frac{\nu-1}{\nu-2} - \lambda\right\} < 2\left\{\frac{\zeta\nu}{\nu-2} + \frac{\nu-1}{\nu-2}\right\} - 1. \tag{6}$$

Due to the conditions imposed on λ, we have that the multiplier of q is non-positive, but the right side of inequality (6) is positive (the proof of this fact is left to a reader), then we have the fulfilment of inequality (6) for $2 \le q < \infty$. Hence (5) holds and using the same reasonings we obtain (2). Taking into account the above reasonings, by virtue of Corollary 1 (see Appendix), we also have the fulfilment of relation (4), if $\lambda > 1/2$. □

The following result is formulated in terms of coefficients of the expansion on $\{e_n\}_0^\infty$ and devoted to conditions under being imposed the inverse operator B^{-1}

exists. Consider the following operator equation under most general assumptions concerning to the right side

$$B\psi = f. \tag{7}$$

Let us chose the Riemann–Liouville fractional integral operator to demonstrate applicability of such consideration. In this case relation (7) provides some kind of generalized Abel equation (see [26]) that becomes the ordinary Abel equation, if we make the well-known assumptions concerning to the right side. Thus, we know if the next conditions hold $I^{1-\alpha}_{a+}f \in AC(\bar{I})$, $\big(I^{1-\alpha}_{a+}f\big)(a) = 0$, $I = (a, b) \subset \mathbb{R}$, then there exists a unique solution of the ordinary Abel equation in the class $L_1(I)$ (see [28]). The sufficient conditions for solvability of equation (7), in the abstract case, are established in the following theorem.

Theorem 2. *Assume that the right side of equation (7) such that the following series converges for arbitrary fixed m, moreover*

$$\sum_{n=0}^{\infty} f_n A'_{mn} = O(m^{-\lambda}),\ \lambda \geq 0, \quad \|A^{-1}S_k f\|_{L_p(\Omega,\mu_1)} \leq C, \quad k \in \mathbb{N}_0. \tag{8}$$

Then there exists a solution of equation (7) in $L_p(\Omega, \mu_1)$, the solution belongs to $L_q(\Omega, \mu_1)$, where $q = p$, if $0 \leq \lambda \leq 1/2$; q is an arbitrary large number satisfying

$$q < \min\left\{\nu, \frac{\nu(2\zeta + 1)}{\nu(\zeta + 1 - \lambda) + 2\lambda - 1}\right\}, \tag{9}$$

if $1/2 < \lambda < [\nu(\zeta + 1) - 1]/(\nu - 2)$; and q is an arbitrary large number satisfying $q < \nu$, if $\lambda \geq [\nu(\zeta + 1) - 1]/(\nu - 2)$. Moreover, if $\lambda > 1/2$, then the solution is represented by a convergent in $L_q(\Omega, \mu_1)$ series

$$\psi = \sum_{m=0}^{\infty} e_m \sum_{n=0}^{\infty} f_n A'_{mn}. \tag{10}$$

Proof. Due to the theorem conditions, we have

$$(A^{-1}S_k f, e_m) \longrightarrow \sum_{n=0}^{\infty} f_n A'_{mn}, \quad k \to \infty,\ m \in \mathbb{N}_0. \tag{11}$$

Since relation (11) holds and the sequence $\{A^{-1}S_k f\}_0^\infty$ is bounded with respect to the norm $L_p(\Omega, \mu_1)$, then due to Theorem 2 [29, p. 216], we have that the sequence $\{A^{-1}S_k f\}_0^\infty$ weakly converges to some function $\psi \in L_p(\Omega, \mu_1)$. Using the ordinary properties of inverse and adjoint operators, taking into account that $A^{-1}S_k f \in L_p(\Omega, \mu_1)$, we obtain the representation

$$\begin{aligned}(S_k f, e_m)_{\mu_1} = (AA^{-1}S_k f, e_m)_{\mu_1} &= (B_0 A^{-1}S_k f, e_m)_{\mu_1} \\ &= (A^{-1}S_k f, B_0^* e_m)_{\mu_1} = (A^{-1}S_k f, B^* e_m)_{\mu_1},\end{aligned}$$

where B_0 is a restriction of B to the set $A^{-1}S_k f$, $k \in \mathbb{N}_0$, thus we have $B_0^* \supset B^*$. Due to the weak convergence of the sequence $\{A^{-1}S_k f\}_0^\infty$, we have

$$(A^{-1}S_k f, B^* e_m)_{\mu_1} \to (\psi, B^* e_m)_{\mu_1} = (B\psi, e_m)_{\mu_1}, \quad k \to \infty.$$

It follows that

$$(S_k f, e_m)_{\mu_1} \to (B\psi, e_m)_{\mu_1}, \quad k \to \infty.$$

Taking into account that

$$(S_k f, e_m)_{\mu_1} = \begin{cases} f_m, & k \geq m, \\ 0, & k < m, \end{cases}$$

we obtain

$$(B\psi, e_m)_{\mu_1} = f_m, \quad m \in \mathbb{N}_0.$$

Using the uniqueness property of a basis expansion, we obtain that the equality $B\psi = f$ holds on the set $\Omega \setminus M$, $\mu_1(M) = 0$. Hence there exists a solution of equation (7). Now let us proceed to the following part of the proof. Note that we have previously proved the fact $\psi \in L_p(\Omega, \mu_1)$, if $0 \leq \lambda < \infty$. Let us show that $\psi \in L_q(\Omega, \mu_1)$, where q is defined by condition (9), if $1/2 < \lambda < [\nu(\zeta+1)-1]/(\nu-2)$. In accordance with the weak convergence established above, we have

$$(A^{-1} S_k f, e_m)_{\mu_1} \longrightarrow (\psi, e_m)_{\mu_1}, \quad k \to \infty, \ m \in \mathbb{N}_0.$$

Combining this fact with (11), we get

$$\psi_m = (\psi, e_m)_{\mu_1} = \sum_{n=0}^{\infty} f_n A'_{mn}, \quad m \in \mathbb{N}_0.$$

Using the theorem conditions (8), we have

$$|\psi_m| \leq C m^{-\lambda}, \quad m \in \mathbb{N}.$$

Now we need to apply Hypothesis 1. For this purpose, let us show that under the assumption $1/2 < \lambda < [\nu(\zeta+1)-1]/(\nu-2)$ relation (5) is fulfilled in terms of this theorem conditions. Having done the reasonings analogous to the reasonings of Theorem 1, we obtain the desired result. Thus we have $\psi \in L_q(\Omega, \mu_1)$, where q is defined by condition (9). The proof corresponding to the case $\lambda \geq [\nu(\zeta+1)-1]/(\nu-2)$ is also analogous to the proof given in Theorem 1. In a similar way, we obtain in this case that $\psi \in L_q(\Omega, \mu_1)$, where q is an arbitrary large number satisfying $q < \nu$. Taking into account the facts given above, by virtue of Corollary 1 (see Appendix) we also have the fulfilment of relation (10), if $\lambda > 1/2$. $\square$

Theorem 3. *Assume that the following conditions hold:* $\mu_1(M) = 0$ *if* $\mu_0(M) = 0$, $M \subset \Omega$; *there exists a sequence of sets*

$$\bigcup_{n=1}^{\infty} \Omega_n = \Omega, \quad \Omega_n \subset \Omega_{n+1},$$

$$L_p(\Omega, \mu_1) \subset L_p(\Omega_n, \mu_0), \quad \mu_0(\Omega \backslash \Omega_n) \to 0, \ n \to \infty,$$

there exist the corresponding sets of functions such that

$$\bigcup_{n=1}^{\infty} \Theta_n = \Theta, \quad \Theta_n \subset \Theta_{n+1},$$

Θ_n, Θ are dense sets in the spaces $L_{p'}(\Omega_n,\mu_0), L_{p'}(\Omega,\mu_0)$ respectively;

$$\forall \eta\in\Theta,\ \exists h\in L_{p'}(\Omega,\mu_1) : (\xi,\eta)_{\mu_0} = (\xi, B^*h)_{\mu_1},\ \xi\in L_p(\Omega,\mu_1). \tag{12}$$

Then a solution of equation (7) *is unique.*

PROOF. Assume that there exists a solution ψ and another solution ϕ in $L_p(\Omega,\mu_1)$ of the Abel equation (7), and denote $\xi:=\psi-\phi$. Due to the theorem conditions, we have

$$(\xi,\eta)_{\mu_0} = (\xi,B^*h)_{\mu_1} = (B\xi,h)_{\mu_1} = 0, \quad \eta\in\Theta,\ h\in L_{p'}(\Omega,\mu_1).$$

Hence

$$(\xi,\eta)_{\mu_0,n} = 0, \quad \forall\eta\in\Theta_n.$$

We claim that $\xi\neq 0$. Therefore, in accordance with the consequence of the Hahn–Banach theorem there exists the element $\vartheta\in L_{p'}(\Omega_n,\mu_0)$, such that

$$(\xi,\vartheta)_{\mu_0,n} = \|\psi-\phi\|_{L_p(\Omega_n,\mu_0)} > 0.$$

On the other hand, there exists the sequence $\{\eta_k\}_1^\infty\subset\Theta_n$, such that $\eta_k\to\vartheta$ with respect to the norm $L_{p'}(\Omega_n,\mu_0)$. Hence

$$0 = (\xi,\eta_k)_{\mu_0,n} \to (\xi,\vartheta)_{\mu_0,n}.$$

Thus, we have come to the contradiction. Hence $\psi=\phi$ on the set Ω_n, $n=1,2,\ldots$. In its own turn, it implies that $\psi=\phi$ on the set $\Omega\setminus M$, $\mu_0(M)=0$. Due to the theorem conditions we conclude that $\psi=\phi$ on the set $\Omega\setminus M$, $\mu_1(M)=0$. The uniqueness has been proved. □

REMARK 1. Using relation (12), we can easily show that

$$(\xi,\eta)_{\mu_0} = (\xi, I\eta)_{\mu_1}, \quad \eta\in\Theta,\ \xi\in L_p(\Omega,\mu_1), \tag{13}$$

where I is a linear operator. Note that due to the made assumptions, we have that $\mathfrak{R}(B)$ is a dense set in $L_p(\Omega,\mu_1)$. Combining this fact with (3.37) [30, p. 24], we conclude that $\ker B^*=0$. Define the semi-norm

$$\begin{aligned}\|\eta\|_{\tilde L_{p'}(\Omega,\mu_1)} &= \sup_{\xi\in L_p(\Omega,\mu_1)} \frac{|(\xi,\eta)_{\mu_0}|}{\|\xi\|_{L_p(\Omega,\mu_1)}} = \sup_{\xi\in L_p(\Omega,\mu_1)} \frac{|(\xi,B^*h)_{\mu_1}|}{\|\xi\|_{L_p(\Omega,\mu_1)}}\\ &= \|B^*h\|_{L_{p'}(\Omega,\mu_1)} = \|I\eta\|_{L_{p'}(\Omega,\mu_1)}, \quad \eta\in\Theta,\ \xi\in L_p(\Omega,\mu_1).\end{aligned} \tag{14}$$

Now if we assume in addition that $\Theta\subset L_p(\Omega,\mu_1)$, then the semi-norm (14) turns into a norm that becomes an ordinary negative norm (see [31]), if the following inequality holds

$$\|\xi\|_{L_p(\Omega,\mu_0)} \le C\|\xi\|_{L_p(\Omega,\mu_1)}, \quad \xi\in L_p(\Omega,\mu_1).$$

Note that the operator I plays an important role in the theory of negative space [31, 32]. Thus the main assumption (12) of Theorem 3 can be reformulated in the following form. There exists an operator I such that (13) holds, moreover

$$I:\Theta\to\mathfrak{D}(B^{*-1}).$$

The last relation can be treated as a characteristic of the uniqueness property for the considered above operator equation in Banach space.

4. Applications

In this section our aim is to justify the application of the obtained abstract results to the fractional integral operator. Throughout this section we use the following notation for the weighted complex Lebesgue spaces $L_p(I,\omega)$, $1 \le p \le \infty$, $\omega(x) = (x-a)^\beta(b-x)^\gamma$, $\beta,\gamma > -1$, where $I = (a,b)$ is an interval of the real axis. If $\omega = 1$, then we use the notation $L_p(I)$. Let us define the left-side fractional integral and derivative of real order $\alpha \in (0,1)$ respectively

$$\big(I^\alpha_{a+}f\big)(x) = \frac{1}{\Gamma(\alpha)}\int\limits_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt, \quad f \in L_1(I),$$

$$\big(D^\alpha_{a+}f\big)(x) = \frac{d}{dx}\big(I^{1-\alpha}_{a+}f\big)(x), \quad f \in I^\alpha_{a+}(L_1),$$

where $I^\alpha_{a+}(L_1)$ is a class of functions represented by the fractional integral defined on $L_1(I)$ (see Definition 2.3 [28, p. 43]). The orthonormal system of the Jacobi polynomials, with the parameters β,γ corresponding to the weighted function, is denoted by

$$p^{\beta,\gamma}_n(x) = \delta_n(x-a)^{-\beta}(b-x)^{-\gamma}\frac{d^n}{dx^n}[(x-a)^{\beta+n}(b-x)^{\gamma+n}], \quad \beta,\gamma > -1,\ n \in \mathbb{N}_0,$$

where

$$\delta_n(\beta,\gamma) = \frac{(-1)^n}{(b-a)^{n+(\beta+\gamma+1)/2}}\sqrt{\frac{(\beta+\gamma+2n+1)\Gamma(\beta+\gamma+n+1)}{n!\Gamma(\beta+n+1)\Gamma(\gamma+n+1)}},$$

$$\delta_0(\beta,\gamma) = \frac{1}{\sqrt{\Gamma(\beta+1)\Gamma(\gamma+1)}}, \quad \beta+\gamma+1 = 0.$$

For the case corresponding to $\beta = \gamma = 0$, we have the Legendre polynomials. Further, we will also use the short-hand notation $p_n := p^{\beta,\gamma}_n$. It is worth noticing that the criterion of the basis property for the Jacobi polynomials was proved by H. Pollard in the paper [33], where the author claims that the Jacobi polynomials have the basic property in the space $L_p(I_0,\omega)$, $I_0 := (-1,1)$, $\beta,\gamma \ge -1/2$, $M(\beta,\gamma) < p < m(\beta,\gamma)$ and do not have the basis property, if $p < M(\beta,\gamma)$ or $p > m(\beta,\gamma)$, where $M(\beta,\gamma)$, $m(\beta,\gamma)$ are constants depending on β,γ. Note that according to the denotations of [26], we have

$$A^{\alpha,\beta,\gamma}_{mn} := \hat{\delta}_m\sum_{k=0}^n(-1)^k\frac{\mathfrak{C}^{(k)}_n(\beta,\gamma)B(\alpha+\beta+k+1,\gamma+m+1)}{\Gamma(k+\alpha-m+1)},$$

where $m,n \in \mathbb{N}_0$, $\alpha \in (-1,1)$, and

$$\mathfrak{C}^{(k)}_n(\beta,\gamma) := (-1)^{n+k}p^{(k)}_n(a)(b-a)^k, \quad \hat{\delta}_m = (-1)^m(b-a)^{m+\alpha+\beta+\gamma+1}\delta_m.$$

Further, we assume that $\beta,\gamma \in [-1/2,1/2]$, $M(\beta,\gamma) < p < m(\beta,\gamma)$. The following coefficients are calculated in [26]:

$$\int\limits_I p_m I^\alpha_{a+}p_n\omega(x)dx = (-1)^n A^{\alpha,\beta,\gamma}_{mn}, \ \int\limits_I p_m D^\alpha_{a+}p_n\omega(x)dx = (-1)^n A^{-\alpha,\beta,\gamma}_{mn},\ \alpha \in (0,1).$$

Let us accept the following notations:

$$I =: \Omega, \quad dx =: d\mu_0, \quad \omega(x)\,dx =: d\mu_1, \quad p_n^{(\beta,\gamma)} =: e_n, \quad I_{a+}^{\alpha} =: A, \quad D_{a+}^{\alpha} =: A^{-1},$$

$$A_{mn}^{\alpha,\beta,\gamma} =: A_{mn}, \quad A_{mn}^{-\alpha,\beta,\gamma} =: A'_{mn}, \quad I_{a+}^{\alpha} =: B, \quad \alpha \in (0,1).$$

Note that

$$\left|p_n^{(\beta,\gamma)}(x)\right| \le C n^{d+1/2}, \quad d = \max\{\beta,\gamma\}, \quad x \in \bar{I}$$

(see [34, p. 288, Theorem 7.3]). Hence, we can define $\nu := \infty$, $\zeta := 1/2 + \max\{\beta,\gamma\}$. In accordance with the results of [26] we conclude that Hypothesis 1 is correct. Thus we can formulate Theorems 1, 2 in terms of the given interpretation (see [30]). Note that the fulfillment of the Theorem 3 conditions does not seem clear. However, to establish them, we can make the following reasonings. Denote

$$\left(a + \frac{1}{n}, b - \frac{1}{n}\right) =: \Omega_n, \quad C_0^{\infty}(\Omega_n) =: \Theta_n, \quad C_0^{\infty}(\Omega) =: \Theta.$$

Then the following assumptions are fulfilled:

$$\bigcup_{n=1}^{\infty} \Omega_n = \Omega, \quad \Omega_n \subset \Omega_{n+1}, \quad \mu_0(\Omega \backslash \Omega_n) \to 0,\ n \to \infty, \quad L_p(\Omega,\mu_1) \subset L_p(\Omega_n,\mu_0).$$

The verification is left to a reader. In terms of these notations, it is also clear that

$$\bigcup_{n=1}^{\infty} \Theta_n = \Theta, \quad \Theta_n \subset \Theta_{n+1},$$

Θ_n, Θ are dense sets in the spaces $L_{p'}(\Omega_n,\mu_0), L_{p'}(\Omega,\mu_0)$ respectively. Let us show that

$$\forall \eta \in \Theta,\ \exists h \in L_{p'}(\Omega,\mu_1) : (\xi,\eta)_{\mu_0} = (\xi, B^* h)_{\mu_1},\ \xi \in L_p(\Omega,\mu_1). \tag{15}$$

It is not hard to prove that $D_{b-}^{\alpha}\eta(x) \in C(\overline{\Omega})$, the proof is left to a reader. Moreover, we have the following. Consider

$$\omega^{-1}(x) D_{b-}^{\alpha}\eta(x) = (x-a)^{-\beta}(b-x)^{1-\gamma}(b-x)^{-1} D_{b-}^{\alpha}\eta(x).$$

Let us show that $D_{b-}^{\alpha}\eta(b) = 0$. Using [28, p. 35, Lemma 2.2], we have

$$\left(D_{b+}^{\alpha}\eta\right)(x) = \frac{1}{\Gamma(1-\alpha)}\left\{\eta(b)(b-x)^{-\alpha} - \int_x^b (t-x)^{-\alpha}\eta'(t)\,dt\right\}$$
$$= -\frac{1}{\Gamma(1-\alpha)}\int_x^b (t-x)^{-\alpha}\eta'(t)\,dt.$$

It is clear that

$$\left|\int_x^b (t-x)^{-\alpha}\eta'(t)\,dt\right| \le C\int_x^b (t-x)^{-\alpha}\,dt = \frac{(b-x)^{1-\alpha}}{1-\alpha}.$$

It gives us the desired result i.e. $D^{\alpha}_{b-}\eta(b)=0$. We can also get without any difficulties: by using the reasonings analogous to [28, p. 32, Lemma 2.1], the following relation:

$$\frac{d}{dx}D^{\alpha}_{b+}\eta(x)=-\frac{1}{\Gamma(1-\alpha)}\int\limits_{x}^{b}(t-x)^{-\alpha}\eta''(t)dt,$$

and by virtue of the previously obtained formula, it is also clear that

$$\frac{d}{dx}D^{\alpha}_{b+}\eta(x)=0,\quad x=b.$$

Having taken into account these facts, we obtain the following:

$$(b-x)^{-1}D^{\alpha}_{b-}\eta(x)=(b-x)^{-1}\big\{D^{\alpha}_{b-}\eta(x)-D^{\alpha}_{b-}\eta(b)\big\}\to 0,\quad x\to b.$$

Hence the function $\omega^{-1}D^{\alpha}_{b-}\eta$ belongs to $L_{p'}(\Omega,\mu_1)$, if $\beta<1/(p'-1)$ (in particularly it is fulfilled if $1<p'\leq 2$). It implies that we have a representation $D^{\alpha}_{b-}\eta=\omega h$, where h belongs to $L_{p'}(\Omega,\mu_1)$. By virtue of the fact $\eta\in C_0^{\infty}(\Omega)$, we can easily prove a relation

$$I^{\alpha}_{b-}D^{\alpha}_{b-}\eta=\eta$$

(see [28, p. 45, formula (2.58)]). Hence

$$\eta=I^{\alpha}_{b-}\omega h,\quad h\in L_{p'}(\Omega,\mu_1).$$

Note that $\omega^{-1}I^{\alpha}_{b-}\omega h$ is a bounded function (it is obvious). Taking into account the reasonings given above, by virtue of the Fubini theorem, we can write

$$\int\limits_{a}^{b}\xi(x)\eta(x)\,dx=\int\limits_{a}^{b}\xi(x)\omega^{-1}(x)I^{\alpha}_{b-}\omega h(x)\omega(x)\,dx=\int\limits_{a}^{b}I^{\alpha}_{a+}\xi(x)h(x)\omega(x)\,dx.$$

Now, it is clear that $\omega^{-1}I^{\alpha}_{b-}\omega h=B^{*}h$ and we conclude that relation (15) holds. As a result, we come to the conclusion that the theoretical results of section 3 have a concrete application to questions connected with the existence and uniqueness theorem for the Abel equation in a weighted case.

5. Conclusions

In this paper, our first aim is to construct an operator model describing the fractional integral action in the weighted Lebesgue spaces. The approach used in the paper is the following: to generalize known results of fractional calculus and in this way to achieve a novel method of studying operators action in Banach spaces. Besides the theoretical results of the paper, we produce the relevance of this direction, which is provided by plenty of applications in various engineering sciences. More precisely tools for a study of such process as electrochemical processes, dielectric polarizations, colored noises are provided by the paragraph application. The foundation of models describing the processes listed above can be obtained by fractional calculus methods, the central point of which is a concept of the Riemann–Liouville operator mapping properties in a weighted Lebesgue space. In its own turn this concept is covered by the theoretical part of this paper. Thus, the obtained results are harmoniously connected with the concrete models of physical and chemical processes.

Appendix

In this section, we formulate the following hypothesis:

Hypothesis 1. *Let $(\Omega, \mathscr{F}, \mu)$ be a space with a finite measure and let $\{e_n\}_1^\infty$ be an arbitrary system of functions orthonormal in $L_2(\Omega, \mu)$. Assume that*

$$\left(\int\limits_\Omega |e_n|^\nu \, d\mu\right)^{1/\nu} \leq M_n, \quad M_n \leq M_{n+1},\ 2 < \nu \leq \infty,$$

and the series

$$\sum_{n=1}^{\infty} |c_n|^q M_n^{\frac{\nu(q-2)}{\nu-2}} n^{\frac{\nu-1}{\nu-2}(q-2)} < \infty, \quad 2 \leq q < \nu,$$

converges. Then there exists a function

$$f \in L_q(\Omega, \mu), \quad \int\limits_\Omega f e_n d\mu = c_n,$$

such that

$$\left(\int\limits_\Omega |f|^q \, d\mu\right)^{1/q} \leq A_{q,\nu} \left(\sum_{n=1}^{\infty} |c_n|^q M_n^{\frac{\nu(q-2)}{\nu-2}} n^{\frac{\nu-1}{\nu-2}(q-2)}\right)^{1/q}. \tag{16}$$

Here $A_{q,\nu}$ depends on q and ν only.

Corollary 1. *Assume that the assumptions of Hypothesis* 1 *hold and Hypothesis* 1 *is correct. Then the following series converges in $L_q(\Omega, \mu)$*

$$f = \sum_{n=0}^{\infty} e_n c_n.$$

The proof follows from (16) in an obvious way. □

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